This is your last written problem set for the course. There are two computer projects after this.

1. Here, you will write a program to construct a reasonably realistic interior structure for an $\sim 8M_\odot$ zero-age main-sequence (ZAMS) star. This is adapted from one of Frank Shu's problems.

- Adopt the equation of state
$$P = (n_i + n_e)\, kT + \frac{1}{3} a T^4, \tag{1}$$
where n_i and n_e are the number densities of atomic nuclei and electrons, respectively. Assume all species are completely ionized, and that each metal atom (carbon, nitrogen, oxygen...) with atomic number A contributes $A/2$ electrons. These assumptions are not perfect for metals but can be largely excused since metals make but a small contribution to the gas pressure.

  Express n_i and n_e, as well as the mean molecular weight μ, as functions of X, Y and Z. For a population I star, a typical set of values might be $X = 0.70, Y = 0.27$, and $Z = 0.03$. So pressure is recast as
$$P = \frac{\rho k T}{\mu m_H} + \frac{1}{3} a T^4, \tag{2}$$
We define a parameter $\beta = P_g/P$, which measures the relative importance of gas pressure.

- For opacity, we will simply take the total opacity to be the sum of the two Rosseland mean opacities from bound-free and electron scattering (HKV Ch. 4). Note there are two problems with this simplification. First, in the deep interior where temperature is high, all electrons are free and there should be no bound-free opacity; second, the true definition of the Rosseland opacity, obtains from the integration of $1/\kappa_\nu$ over all frequencies, weighted by the blackbody spectrum (eq. 4.22 of HKV). These two problems, however, are not severe as they seem. In the deep interior, electron scattering is more important; the sum of the two Rosseland means asymptotes to one or the other terms if the two terms alternately dominates at different parts of the star. And it save us having to look up the opacity table.

- The convective core. Given the high core density and temperature, convection is very efficient. We can take the approximation that entropy is constant there. Consult HKV §3.7 to obtain $(d\ln T/d\ln P)_{\rm ad}$ for a mixture of ideal gas and radiation.

- For nuclear generation, include both energy generation rates from PP and CNO burning (see HKV Ch. 6).

Now we are armed with all the physics we need. Go ahead and write an integration code.

- First we have to choose an independent variable to integrate along. I would suggest using $\log P$. Contemplate why this may be superior to using, say, radius, mass ($M(r)$), or linear pressure. You might also want to adopt $\log\rho$ and $\log T$ as variables, as opposed to using ρ and T.

- Guess a central temperature and central pressure, and integrate outward the equations of stellar structure. Assume the star is convective in the interior, until where the radiative temperature gradient (the value of $d\ln T/d\ln P$ that is sufficient to transport the nuclear luminosity outward by radiative diffusion) falls below your calculated $(d\ln T/d\ln P)_{\rm ad}$. Once this occurs, switch over to radiative flux transport and continue to integrate towards the surface.

- The radiative photosphere is defined at a pressure

$$P_{\rm ph} = \frac{2}{3}\frac{g}{\kappa}. \tag{3}$$

  You can obtain this relation by combining the equations of hydrostatic equilibrium and radiative transfer, and by remembering the definition of the photosphere (optical depth $\tau = 2/3$). This pressure expression can be translated into $P_{\rm ph} = P_{\rm ph}(T = T_{\rm eff})$, where the effective temperature is related to the nuclear luminosity as

$$L = 4\pi R^2 \sigma T_{\rm eff}^4. \tag{4}$$

- A duet of initial guesses suggested by Frank Shu are: $T_c = 2.88 \times 10^7$ K, and $P_c = 4.02 \times 10^16$ dyne/cm^2. These guesses, when integrated outward, may not necessarily intersect the $P_{\rm ph} = P_{\rm ph}(T = T_{\rm eff})$ you obtain above. Keep T_c fixed, and vary P_c to minimize, e.g.,$|\log P - \log P_{\rm ph}|$. You may use bisection or other root finding technique to find the relevant P_c (consult "Numerical Recipes" by Press et al).

- Display your results by plotting $\log T$, $\log \rho$, radius, M, and luminosity, as functions of $\log P$.

- The stellar mass you get this way may not be exactly $8M_\odot$, but you can see that if you fine-tune T_c, you will get there. If the mass you get above is higher than $8M_\odot$, should you raise or lower your guess for T_c?

Now we go on to analyze your computer output. Discuss

- relative importance of the PP-chain vs. the CNO-cycle in an $8M_\odot$ star.

- relative importance of the bound-free, free-free and electron scattering opacity in such a star.

- relative importance of radiation pressure here.

- why a main-sequence star of intermediate (or high) mass has a convective core. Pay attention to both the run of $(d\ln T/d\ln P)_{\rm ad}$ with β, as well as to the radial dependence of the production of luminosity inside the core.

- why do main-sequence stars of intermediate and high mass have radiative envelopes? Compare the radiative temperature gradient ($d\ln T/d\ln P$ for radiative diffusion) against the adiabatic temperature gradient. What do you think would occur for more massive, and therefore, more luminous and more radiation pressure dominated stars?

2. Structure of a shell-source star (a giant).

Evolved stars often encounter a situation in which the inner dense core has finished nuclear burning, and is largely isothermal[1], while the outer envelope expands in response to a thin nuclear burning shell immediately outside the core. We call these stars the red giants if the star has a pure helium core and burns hydrogen in the thin shell, or asymptotic giants if the core is carbon-oxygen and helium (and sometimes hydrogen) burning occurs in the shell.

For a low mass star, the structure of a shell star differs from that of a white dwarf (our last problem set) only in that the non-degenerate envelope is not geometrically thin. For a high mass star, radiation pressure is significant in supporting the core and the shell.

Here, we (re-)derive homology relations for both low and high mass giants. Much of the materials here are covered in: KW §19.8, 30.5, 32.2 & 33.5, and their supporting sections. I attempt to combine them all into the same problem, to highlight the similarities and contrasts.

Let the core temperature be T_c, mass M_c, radius R_c, pressure at the core boundary be P_b; the total stellar mass M, total radius R, and luminosity (from the shell source) L. Hydrostatic and thermal equilibrium require that pressure and temperature at the core-envelope interface be continuous.

- The Schonberg-Chandrasehkar limit. For a core that is supported by ideal gas pressure, and only differs from the envelope in its mean molecular weight, there is a maximum value for the fractional mass inside the core (Schonberg-Chandrasekhar, 1942). Here, we illustrate this mass limit by combining numerical integration and analytical scaling.

  Consider an isothermal core supported by ideal gas pressure, with a finite pressure at its surface (due to the weight of the envelope).[2] Express pressure as

$$P = \frac{\rho k T_c}{\mu_c m_H} = \rho c_c^2, \tag{5}$$

  where μ_c is the mean molecular weight of the core, and square of the core sound speed $c_c^2 = kT_c/\mu_c m_H$. Follow §19.8 of KW to obtain the isothermal Lane-Emden equation.

  In order to integrate this equation, we will need an asymptotic expression for $\omega \equiv \Phi/c_c^2$ at $z \ll 1$ (notation here follows that of KW). Show that the following power series

$$\omega \approx \frac{z^2}{6} + \mathcal{O}(z^4) + ... \tag{6}$$

  satisfies the Lane-Emden equation for $\omega \ll 1$ and $z \ll 1$. Numerically integrate the second-order Lane-Emden equation, using the above initial condition, to obtain ω as a function of z. Plot your density as a function of z, and show that the density can be approximated by two different relations,

$$\begin{aligned} \rho &\sim \rho_c & z = Ar \ll 1, \\ &\sim \rho_c z^{-\alpha} & z = Ar \gg 1, \end{aligned} \tag{7}$$

[1] No flux goes in or out so a temperature gradient is not maintained

[2] This is akin to the Bonner-Ebert sphere in star formation (KW §26.2)

where ρ_c is the central density. Obtain the value for α. If we define a scale radius $r_s = 1/A$, this says that the density is roughly constant within the scale radius, but falls off as a power-law outside of it.

Obtain the total core mass for the two separate regimes, $R_c \ll r_s$ or $R_c \gg r_s$. In the latter regime, core mass rises with core radius, and there is no bound to the mass, an interesting property of the isothermal sphere.

Express the core pressure at R_c, P_-, for both regimes as functions of the core mass, as well as the core sound speed.

Now consider the envelope. Assume the core exerts a sufficiently small influence that the envelope follows largely that of a homologous polytrope model. Obtain the following scaling for the base pressure as a function of the base sound speed (c_e),

$$P_+ \sim \frac{c_e^8}{G^3 M^2}, \tag{8}$$

where M is now the total mass of the star.

Balancing P_+ and P_- to obtain the following scaling

$$\frac{M_c}{M} \leq \left(\frac{\mu_e}{\mu_c}\right)^2 \tag{9}$$

an equation that differs only by a factor of unity from eq. (30.14) of KW. Argue why this is an upper limit to the core mass.

This is the Schonberg-Chandrasekhar mass limit. Once the core mass exceeds this limit ($M_c/M \sim 0.1$ for a helium core), the core is forced to contract into a different state, with gravity supported by either electron degeneracy (when T_c is low, as is appropriate for a low mass star), or by radiation pressure (when T_c is high, as is appropriate for a high mass star). These are dealt with below.

- The core mass-luminosity relation for a low mass star (Refsdal & Weigert, A& A, 1970, 2, 426). Consider a core of non-relativistically degenerate helium surrounded by an extended envelope of unburnt material, which is in turn supported by ideal gas pressure. If hydrogen burning is occurring in a thin shell above the core, it is said that the luminosity of the shell source depends uniquely on the core mass, with $L \propto M_c^\alpha$ and α being a large positive number. As a result, as the post-main-sequence Sun ages and accumulates in core mass, its luminosity rises.

  Density in the degenerate core can be orders of magnitude higher than that in layer immediately above. One can therefore safely assume that the layer above contributes negligibly to the mass and gravity (until much beyond the burning zone). The star radius $R \gg R_c$.

  We are going to follow Refsdal & Weigert (1970) in adopting the following homology solutions for the layer immediately above the core. Let $z = r/R_c \geq 1$ (but z never reaches much greater than unity) , they are

$$\begin{aligned} P &= p'(z) M_c^{\alpha_p} R_c^{\beta_p} \\ T &= T'(z) M_c^{\alpha_T} R_c^{\beta_T} \\ L &= L'(z) M_c^{\alpha_L} R_c^{\beta_L}. \end{aligned} \tag{10}$$

All models with different M_c and R_c share the same $p'(z), T'(z)$ and $L'(z)$. These scaling conveniently provide us with the following analytical results – but why are they reasonable? We will come back to this later.

Substituting these homology relations into the equation for hydrostatic equilibrium to obtain

$$\alpha_T = 1, \qquad \beta_T = -1. \tag{11}$$

Continue with this exercise and obtain the exponents for luminosity, by inserting the homology relations into the equations of radiative transfer and nuclear generation, in particular, adopt the electron scattering opacity and the CNO cycle, with

$$\epsilon = \epsilon_0 \rho T^{13}. \tag{12}$$

In the last step we adopt the dependence of R_c on M_c as that for a cold white dwarf. With your final result for $L \propto M_c^{\alpha}$, can you answer the following question: .....

Now let us return to the question of the homology relation in equation 10. Instead of the reduced version there, let us make sure we are not forgetting any physical variable and adopt the following set:

$$\begin{aligned} P &= p'(z) M_c^{\alpha_p} R_c^{\beta_p} M^{\gamma_p} R^{\delta_p} \\ T &= T'(z) M_c^{\alpha_T} R_c^{\beta_T} M^{\gamma_T} R^{\delta_T} \\ L &= L'(z) M_c^{\alpha_L} R_c^{\beta_L} M^{\gamma_L} R^{\delta_L} \end{aligned} \tag{13}$$

Follow the same exercise as before to show that all γ and δ are necessarily zero. Or, global stellar properties are not very important in the core mass-luminosity relation. Now after the math, can you qualitatively explain – with a target audience of astronomy graduate students – why giant luminosity depends almost exclusively only on core mass?

- The core mass-luminosity relation for a high mass star (Paczynski, 1970, Acta. Astron. 20, 47). We consider the extreme case when the pressure support is dominated by radiation pressure. Show that the luminosity production of the star scales linearly with the core mass, with the luminosity being the Eddington luminosity for the core mass.[3]

- Now... if I was feeling more energetic, I would have asked you the question of why the transition from $L \propto M^{\alpha}$ (low mass core) to $L \propto M$ (high mass core) occurs around $M_c = M_{\rm ch}$, the Chandrasekhar mass. But, I run out of steam.

[3]The usual Eddington luminosity is defined for electron scattering opacity, but it can be easily generalized.