6. Plasma Effects

6.1 Dispersion

Consider an isotropic and homogeneous plasma. Strictly speaking, a plasma can have charge and current densities and so the "source-free" form of Maxwell's equations no longer applies. However, if there are no *external* sources, the equations *can* be expressed in the source-free form with the polarizability of the medium incorporated into the dielectric constant ϵ. That is, the medium is neutral overall but it can be polarized by applying an electric field or sending an EM-wave through it. The passage of an EM-wave (for example) through such a plasma induces a collective motion of the charged particles, thereby changing the propagation properties (namely, the phase velocity and the index of refraction).

From Chandrasekhar's *Plasma Physics* or p52 of the text, Maxwell's equations for a dielectric medium with no magnetic permeability ($\mu = 1$) can be written in the form

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= 0\,, & \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0\,, \\ \boldsymbol{\nabla} \times \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}\,, & \boldsymbol{\nabla} \times \boldsymbol{B} &= \frac{\epsilon}{c}\frac{\partial \boldsymbol{E}}{\partial t}\,, \end{aligned}$$

where $\epsilon = 1 + 4\pi\alpha$. Here, α is the polarizability defined by $\boldsymbol{P} = \alpha\boldsymbol{E}$ where $\boldsymbol{P}$ is the dipole moment per unit volume. If we were to derive the wave equations for $\boldsymbol{E}$ and $\boldsymbol{B}$ from the above equations, we would find that $v_{\rm ph} = c/\sqrt{\epsilon}$. To obtain ϵ, we evaluate α for a wave passing through the plasma:

$$m\ddot{\boldsymbol{r}} = e\boldsymbol{E} = e\boldsymbol{E}_o\, \mathrm{e}^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}\,.$$

In writing this, we are assuming that thermal motions are not important. That is, the acceleration of electrons due to collisions with ions is neglected compared to the acceleration induced by $\boldsymbol{E}$. Such collisions can lead to the attenuation or absorption of EM waves at very low frequencies.

Solving the equation of motion, we find

$$\boldsymbol{r} = -\frac{e}{m\omega^2}\,\boldsymbol{E}\,.$$

In the limit of long wavelengths ($\lambda \gg N_e^{-1/3}$), the induced dipole moment is $\boldsymbol{P} = Ne\boldsymbol{r}$ which gives us

$$\boldsymbol{P} = -\frac{Ne^2}{m\omega^2}\,\boldsymbol{E} \quad \Longrightarrow \quad \alpha = -\frac{Ne^2}{m\omega^2} \qquad (N \equiv N_e)$$

and

$$\epsilon = 1 - \frac{4\pi Ne^2}{m\omega^2}\,.$$

Introduce the plasma frequency $\omega_p^2 \equiv 4\pi Ne^2/m$ which yields

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}\,.$$

This obviously makes $v_{\rm ph} > c$ in a tenuous plasma but the index of refraction is *less* than unity:

$$n = \frac{c}{v_{\rm ph}} = \epsilon^{1/2} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}\,.$$

NOTES:

(a) The polarization of an ionized medium is quite different from that of a neutral gas or fluid where the electrons are *bound*. More specifically, in a neutral fluid such as water, $v_{\rm ph} < c$ and $n > 1$ due to a phase shift in the electron's displacement (see p44).

(b) Waves with $\omega < \omega_p$ do not propagate through the plasma. Part of the energy is absorbed and the remainder is reflected because $k = \omega/v_{\rm ph}$ is *imaginary*. This is why AM radio waves "bounce" off the Earth's ionosphere.

(c) The departure of n from unity and $v_{\rm ph}$ from c arises from the collective motion of the charged particles as the EM-wave passes through the medium. This can take place only if the wavelength is greater than the particle separation; i.e., only if $\lambda \gg N^{-1/3}$. If this condition is not satisfied, the medium cannot be polarized, leaving $\alpha = 0$, $\epsilon = n = 1$ and $v_{\rm ph} = c$. In the ISM, the electron separation is on the order of a few cm which means that dispersion is important only for radio wavelengths.

(d) Numerically, the plasma frequency has the value

$$\omega_p = 5.63 \times 10^4 \, N_e^{1/2} \text{ Hz}$$

which is about 10 MHz for the Earth's ionosphere where $N_e \sim 10^4 - 10^6 \, \text{cm}^{-3}$. This falls between the AM and FM bands explaining why we can generally hear AM stations farther away than FM ones. In the ISM, on the other hand, the corresponding plasma frequency is only about 1 KHz due to the small electron density.

(e) The phase velocity is larger than c but, of course, the group velocity must be less than the velocity of light:

$$k = \frac{\omega}{v_{\rm ph}} = \frac{\omega}{c}.\frac{c}{v_{\rm ph}} = \frac{\omega n}{c}$$

or

$$k = \frac{\omega}{c}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} = \frac{1}{c}(\omega^2 - \omega_p^2)^{1/2} \, .$$

From this dispersion relation, we can compute the *group* velocity:

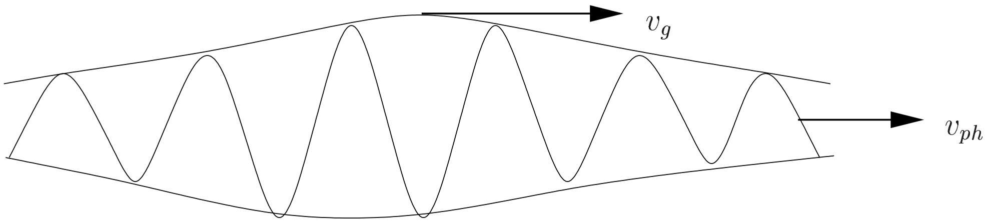
$$v_g = \frac{\partial \omega}{\partial k} = \left(\frac{\partial k}{\partial \omega}\right)^{-1} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} = cn < c \, .$$

Compare this with

$$v_{\rm ph} = \frac{\omega}{k} = \frac{c}{n} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2} > c \, .$$

Thus, we see that $v_g v_{\rm ph} = c^2$.

The group velocity is an important concept. Information is added to a "carrier" wave by *modulating* the amplitude or the frequency of the carrier. The modulation envelope propagates at the "group" velocity.



An important application of the group velocity formula is to the pulses we observe from pulsars. Each pulse has a spectrum which covers a broad range of frequencies each of which propagates at a different group velocity. Therefore, the pulse is *dispersed* by the ionized component of the ISM.

The time required for a pulse to reach the Earth is

$$t_p = \int_0^d \frac{ds}{v_g} = \frac{1}{c}\int_0^d \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2} ds \ .$$

Now for the ISM, $\omega \gg \omega_p$ for frequencies of interest due to the low electron density. Therefore,

$$\begin{aligned} t_p &= \frac{1}{c}\int_0^d ds + \frac{1}{c}\int_0^d \frac{\omega_p^2}{2\omega^2}\, ds \\ &= \frac{d}{c} + \frac{1}{2c\omega^2}\int_0^d \omega_p^2 \, ds \ . \end{aligned}$$

The second term on the RHS side of this equation is obviously the delay in the signal introduced by the dispersive medium. Substituting for ω_p, we have

$$t_p = \frac{d}{c} + \frac{1}{2c\omega^2}\left(\frac{4\pi e^2}{m}\right)\int_0^d N_e \, ds = \frac{d}{c} + \frac{2\pi e^2}{mc\omega^2}\int_0^d N_e \, ds \ .$$

The arrival time is clearly frequency dependent. A plot of t_p versus ω^{-2} yields a straight line whose slope gives the "dispersion measure" D_{M}:

$$D_{\mathrm{M}} \equiv \int_0^d N_e \, ds \qquad \text{(a column density)} \ .$$

We can use the canonical figure of $N_e = 0.03\,\mathrm{cm}^{-3}$ for the ISM to infer pulsar distances. Or, if we have an independent estimate of the pulsar's distance d, we can use the observed dispersion measure to get information on the distribution of mass in the galactic disk or, at least, a reading on the mean electron density $\langle N_e \rangle$ along the line of sight. In fact, it is this technique more than any other that has provided us with the value of $0.03\,\mathrm{cm}^{-3}$.

To get a feel for the magnitude of the dispersion due to the ISM, consider a frequency of 1 GHz ($\omega = 2\pi$ GHz), an electron density of $N_e \sim 0.03\,\mathrm{cm}^{-3}$, and a distance of 1 Kpc. The corresponding time delay relative to vacuum propagation is

$$t_{\mathrm{delay}} = \frac{2\pi e^2}{mc\omega^2}\langle N_e \rangle \, d \sim \frac{1}{10}\,\mathrm{sec} \ .$$

6.2 Faraday Rotation

A field $\boldsymbol{B}$ threading a plasma also affects the propagation properties of the medium; that is, like a nonzero charge density, it affects the dielectric constant ϵ and the index of refraction n. To find the new ϵ, we have to solve a modified equation of motion:

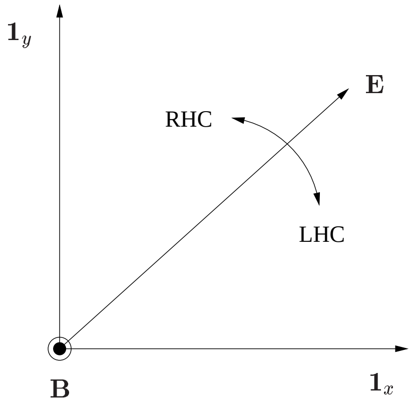
$$m\ddot{\boldsymbol{r}} = e\boldsymbol{E} + \frac{e}{c}(\boldsymbol{v} \times \boldsymbol{B}) \ .$$

Note that we neglect the $\boldsymbol{B}$-field of the EM-wave because its effect is negligible for nonrelativistic electrons. That is, the $\boldsymbol{B}$ in the foregoing equation is the galactic magnetic field (say) and not the field of the EM-wave.

Consider *circularly* polarized waves propagating along the z-axis:

$$\begin{aligned} \boldsymbol{E} &= E(\mathbf{1}_x \pm i\mathbf{1}_y) \\ \boldsymbol{r} &= r(\mathbf{1}_x \pm i\mathbf{1}_y) \end{aligned}$$

where $E = E_0 \mathrm{e}^{i\omega t}$ and $r = r_0 \mathrm{e}^{i\omega t}$. Note that the top and bottom signs in these relations indicate left-handed and right-handed circular polarization, respectively, if the propagation is out of the page. Keep these associations in mind as you study the following analysis.



Now

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = i\omega\boldsymbol{r} \ .$$

Therefore,

$$\boldsymbol{v} \times \boldsymbol{B} = i\omega(\boldsymbol{r} \times \boldsymbol{B}) \ ,$$

or

$$\begin{aligned} \boldsymbol{v} \times \boldsymbol{B} &= i\omega r B(\mathbf{1}_x \pm i\mathbf{1}_y) \times \mathbf{1}_z \\ &= i\omega r B(-\mathbf{1}_y \pm i\mathbf{1}_x) \\ &= \omega r B(\mp\mathbf{1}_x - i\mathbf{1}_y) \\ &= \mp\omega r B(\mathbf{1}_x \pm i\mathbf{1}_y) \ . \end{aligned}$$

Putting it all together, we can drop the common factor $(\mathbf{1}_x \pm i\mathbf{1}_y)$ to obtain

$$-m\omega^2 r = eE \mp \frac{e}{c}\omega B r \ ,$$

or

$$\boldsymbol{r} = -\frac{e\boldsymbol{E}}{m(\omega^2 \mp \omega\omega_{\mathrm{B}})}$$

where $\omega_{\mathrm{B}} \equiv eB/mc$ is the gyromagnetic frequency. Note that the top (minus) sign still corresponds to LHC polarization and the bottom (plus) sign to RHC polarization.

By analogy with the $\boldsymbol{B} = 0$ case where $\boldsymbol{r} = -e\boldsymbol{E}/m\omega^2$, we can immediately write down the dielectric constant. It is

$$\epsilon = 1 - \frac{\omega_p^2}{(\omega^2 \mp \omega\omega_{\mathrm{B}})}$$

and

$$n = \epsilon^{1/2} = \left(1 - \frac{\omega_p^2}{(\omega^2 \mp \omega\omega_{\rm B})}\right)^{1/2} .$$

Also, we have

$$k = \frac{\omega}{v_{\rm ph}} = \frac{\omega n}{c} = \frac{\omega}{c}\left(1 - \frac{\omega_p^2}{(\omega^2 \mp \omega\omega_{\rm B})}\right)^{1/2} .$$

Thus, a LHC polarized wave propagates at a different rate than a RHC polarized one. Normally, $\omega \gg \omega_p$ and $\omega \gg \omega_{\rm B}$. Therefore,

$$k \simeq \frac{\omega}{c}\left[1 - \frac{\omega_p^2}{2\omega^2}\left(1 \pm \frac{\omega_{\rm B}}{\omega}\right)\right] .$$

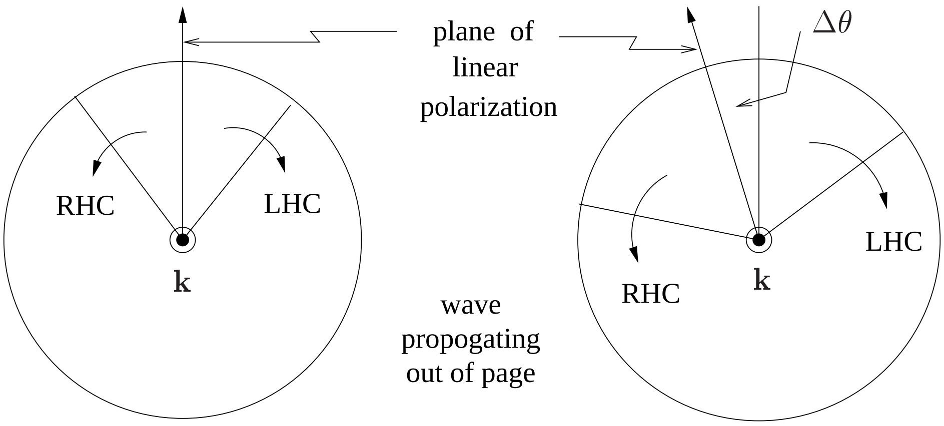
We now ask what happens to a *linearly* polarized wave as it propagates along the field lines. Such a wave is a linear superposition of RHC and LHC polarized waves which, as we have just seen, propagate at different speeds. Therefore, the plane of polarization of the composite wave will *rotate* as it propagates. This is called "Faraday rotation".

The phase angle through which the electric vector of a circularly polarized wave moves in travelling a distance $\boldsymbol{d}$ is just $\boldsymbol{k}\cdot\boldsymbol{d}$. More generally, if the wave vector is not constant along the path, then the changes in the phase angle for the two waves are

$$\phi_{\rm R,L} = \int_0^d k_{\rm R,L}\, ds .$$

The change in the position angle of the plane polarized wave is $\frac{1}{2}$ the difference of the changes in the phase angles of the RHC and LHC waves; i.e.,

$$\Delta\theta = \frac{1}{2}\int_0^d (k_{\rm R} - k_{\rm L})\, ds .$$



(a) $t = 0,\ s = 0$ (b) $t > 0,\ s = d$

But,

$$k_{\rm R} - k_{\rm L} = \frac{\omega}{c} \times \frac{\omega_p^2}{\omega^2} \times \frac{\omega_{\rm B}}{\omega} = \frac{\omega_p^2\omega_{\rm B}}{c\omega^2} .$$

Therefore,

$$\Delta\theta = \frac{1}{2}\int_0^d \frac{\omega_p^2\omega_{\mathrm{B}}}{c\omega^2}\,ds = \frac{2\pi e^3}{m^2c^2\omega^2}\int_0^d N_e\,B\,ds\ ,$$

where I have substituted for ω_p and ω_{B}. This formula is valid only if $\boldsymbol{B}$ lies along the line of sight. However, we can use the formula if we interpret B as $B_{||}$, the component along the line of sight.

Observations of Faraday rotation as a function of frequency (again in the pulses from pulsars) give us information on the "rotation measure"

$$R_{\mathrm{M}} \equiv \int_0^d N_e\,B\,ds\ .$$

Clearly, the dispersion measure D_{M} combined with the rotation measure tells us something about the galactic field strength. Such observations over the years have yielded $\langle B_{\mathrm{gal}}\rangle \sim 3\times 10^{-6}$ gauss.

In interpreting the data on $\Delta\theta$, there is an obvious problem associated with the modularity of angles. That is $\Delta\theta$ could be $\Delta\theta + n\pi$ where n is an unknown integer. However, in practice, this is normally not a problem within our own galaxy because the rotation of the plane of polarization locally is usually less than 2π radians. As an example, consider a frequency of 1 GHz, a density of $N_e = 0.03\,\mathrm{cm}^{-3}$, a field strength of $\langle B_{\mathrm{gal}}\rangle$, and a distance of 1 Kpc. In this situation,

$$\Delta\theta \simeq 1 \text{ radian} \simeq 60^\circ\ .$$

For extragalactic sources, one must make an intelligent guess concerning the density along the line of sight, both within our own galaxy and within the source system, so as to make sure that we use the correct n. Of some help is the knowledge that $\Delta\theta \to 0$ at very high frequencies.

6.3 Cherenkov Radiation

We have learned that a particle in uniform motion in a vacuum does not radiate. However, if the particle velocity *exceeds* the phase velocity of light, this rule is no longer valid; there will be radiation! This phenomenon requires the index of refraction to be *larger* than unity which restricts it to neutral gases and fluids, at least as far as EM-waves are concerned. Other types of waves (such as longitudinal plasma waves) can experience the Cherenkov effect in an ionized gas. Shock waves in the Earth's atmosphere induced by supersonic aircraft is another (acoustic) analog of the Cherenkov effect which does not involve EM-waves.

Recall that the $\boldsymbol{E}$-field of a uniformly moving particle looks like

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{e(\boldsymbol{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3 R^2}$$

where in the current situation

$$\kappa = 1 - \frac{nv}{c}\cos\theta \ ,$$

θ being the angle between $\boldsymbol{n}$ and $\boldsymbol{\beta}$ at the retarded time. In a vacuum, $n = 1$ and κ can become very small (but never zero) for relativistic particles moving in the $\boldsymbol{n}$ direction (toward the observer). This gives rise to the strong beaming of synchrotron radiation. [Warning: the scalar n and the vector $\boldsymbol{n}$ in the foregoing equations are very different quantities!]

Now consider what happens when the particle speed exceeds the phase speed c/n which is possible if $n > 1$. In this case, κ can actually go to zero (!) at a very particular angle θ_o given by

$$\cos\theta_o = \frac{c}{nv} \ .$$

In this direction, the Liénard-Wiechert potentials become *infinite*, invalidating the R^{-2} dependence of the $\boldsymbol{E}$-field. As a consequence, the uniformly moving particle can now radiate in the θ_o-direction and the radiation is strongly beamed into a hollow cone pattern with azimuthal symmetry about the velocity vector.

The physical interpretation is that there is a "shock wave" in the dielectric medium similar or analogous to that which accompanies supersonic motion. The diagrams on the next page (taken from the text, p234) illustrate the shock front and the Cherenkov cone. The conical surface is tangent to all the spheres describing the surfaces of the "potential wave" travelling at velocity c/n. Note that within the Cherenkov cone, each point is intersected by two spheres and thus feels the potentials due to *two* retarded positions of the particle. The Cherenkov radiation moves outwards in a direction *normal* to the cone's surface.

Some particle detectors (such as those used by the Solar Neutrino Observatory in Sudbury) take advantage of the Cherenkov phenomenon. The precise direction of the radiation can be used as an energy measurement for fast particles in the laboratory or observatory. Cherenkov radiation has also been observed from cosmic rays entering the Earth's atmosphere. As far as I know, there are no extraterrestrial detections of Cherenkov emission although, in principle, cosmic rays moving through neutral gas should emit such radiation. The intensity, however, is very low – much lower than that in the form of synchrotron radiation.

THE CHERENKOV CONE

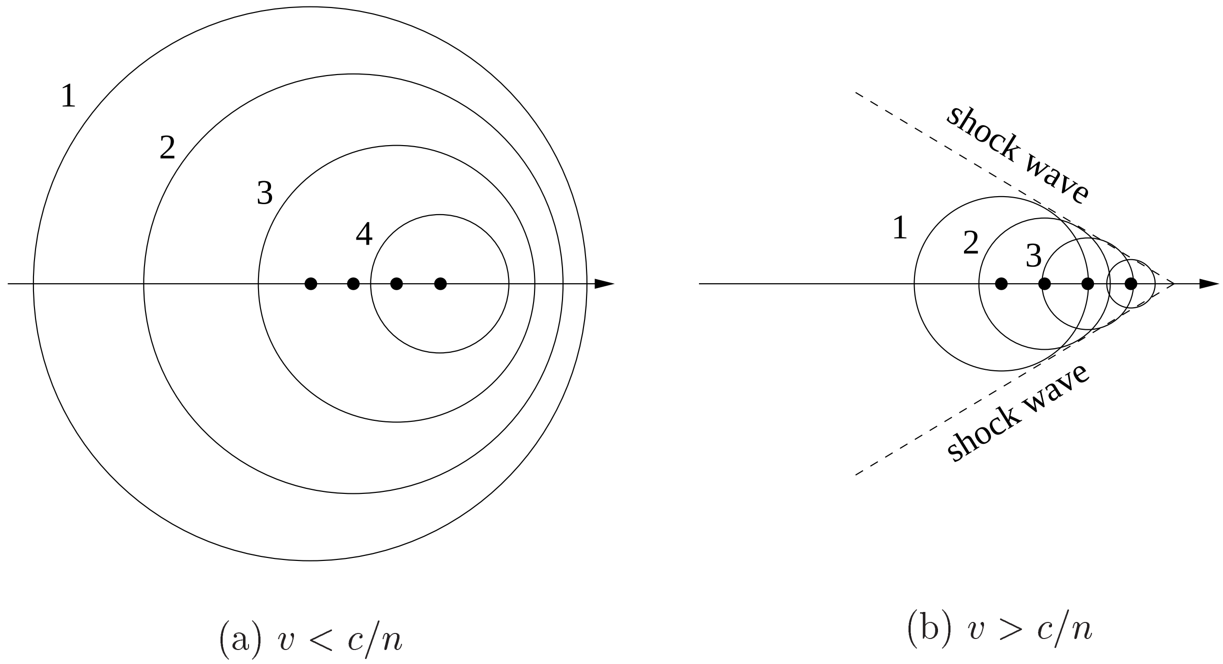


Figure 6.31 – Propagation of wave fronts generated by a particle moving with velocity v through a refractive medium.

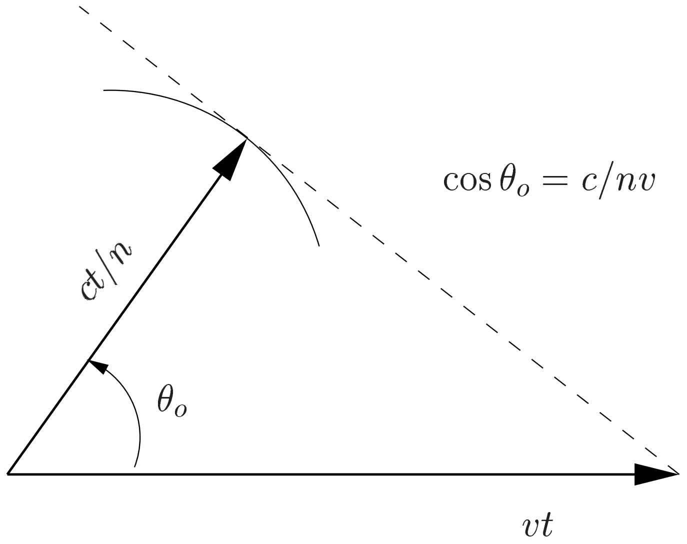


Figure 6.32 – The geometry of the Cherenkov cone in the case $c < nv$.

6.4 The Razin Effect

This phenomenon depends on the index of refraction being *less* than unity which we know occurs in all ionized plasmas, particularly at low frequencies. A consequence is the "quenching" or damping of synchrotron radiation at low frequencies. To see this, consider the beaming effect associated with relativistic electrons:

$$\theta_b \sim \frac{1}{\gamma} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \qquad \text{(in a vacuum)}\,.$$

In a plasma, on the other hand, c gets replaced by c/n and so

$$\theta_b \sim \left(1 - \frac{n^2 v^2}{c^2}\right)^{1/2} \equiv \frac{1}{\gamma_{\text{eff}}}$$

where

$$n^2 = 1 - \frac{\omega_p^2}{\omega^2}\,.$$

Clearly, for sufficiently small ω, the beaming effect will disappear! We have

$$\theta_b = \left[1 - \frac{v^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\right]^{1/2} \simeq \left(\frac{1}{\gamma^2} + \frac{\omega_p^2}{\omega^2}\right)^{1/2}$$

since $v/c \sim 1$. It's obvious now that when $\omega \ll \gamma\omega_p$, the synchrotron spectrum will be cutoff because of the suppression of beaming. This is the Razin effect which dominates the ordinary plasma cutoff that occurs when $\omega < \omega_p$. Once beaming disappears so do the high frequencies associated with synchrotron radiation. Recall that $\nu_c = \frac{3}{2}\nu_{\text{B}}\gamma_{\text{eff}}^2$ and so the low frequency radiation will not be "boosted" by as large a factor as γ_{eff} falls off. This will result in a turnover in the synchrotron spectrum at some low frequency.

As far as I know, this effect has not yet been observed partly because one has to distinguish it from ordinary self-absorption which also leads to a turnover in the spectrum. In galactic sources such as SN remnants, $\gamma\nu_p$ will be higher than the self-absorption turnover (e.g., 10-100 MHz versus 1 MHz) but still somewhat below the radio astronomy waveband.

In dense sources such as quasars, $\gamma\nu_p$ will be higher ($\sim$1 GHz) and closer to the self-absorption turnover. The Razin quenching may still be detectable, however, if we make a bigger effort to model the density distribution in these sources and if we look harder at the evidence. For example, it has been suggested that there is a correlation between radio quiet quasars and high electron densities. If so, perhaps the Razin effect is responsible.

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