# II. CONTINUUM PROCESSES INVOLVING FREE ELECTRONS

## 2. Classical Radiation Theory

### 2.1 Single Charged Particle

**Retarded Potentials**

It is simpler in the final analysis to deal with *potentials* rather than *fields*, particularly since one of them is a scalar. That is, we can reduce the problem from six unknowns (3 components each of $\boldsymbol{E}$ and $\boldsymbol{B}$) to four.

(a) *Vector Potential* $\boldsymbol{A}$:

Since

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \quad \text{and} \quad \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0 \ ,$$

it suggests that we express $\boldsymbol{B}$ as the *curl* of a vector potential; call it $\boldsymbol{A}$: $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$.

(b) *Scalar Potential* $\phi$:

From Maxwell's equations, we have

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{1}{c}\left(\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{A}}{\partial t}\right) \ ;$$

that is,

$$\boldsymbol{\nabla} \times \left(\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\right) = 0 \ .$$

Since $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = 0$, we can express

$$\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{\nabla}\phi$$

where $\phi$ is a scalar potential.

Note that $\boldsymbol{A}$ and $\phi$ are not unique; we can replace $\boldsymbol{A}$ with $\boldsymbol{A} + \boldsymbol{\nabla}\psi$ and $\phi$ by $\phi - \frac{1}{c}\frac{\partial \psi}{\partial t}$ and leave $\boldsymbol{E}$ and $\boldsymbol{B}$ unchanged. We are free then to choose a $\psi$ which simplifies the equations. The standard choice is the "Lorentz Gauge" which satisfies

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0 \qquad \text{(Lorentz condition)}$$

and transforms the remaining Maxwell equations into (see *Radiation Processes*, p71)

$$\begin{aligned} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} &= -4\pi\rho \ , \\ \nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} &= -\frac{4\pi}{c}\boldsymbol{j} \ . \end{aligned}$$

These equations have integral solutions:

$$\begin{aligned}\phi(\boldsymbol{r},t) &= \int \frac{\rho(\boldsymbol{r}',t')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,d^3r' \;, \\ \boldsymbol{A}(\boldsymbol{r},t) &= \int \frac{\boldsymbol{j}(\boldsymbol{r}',t')}{c|\boldsymbol{r}-\boldsymbol{r}'|}\,d^3r' \;,\end{aligned}$$

where $t'$ is the *retarded* time

$$t' \equiv t - \frac{1}{c}|\boldsymbol{r}-\boldsymbol{r}'| \;.$$

Refer to *Classical Electrodynamics* by J. D. Jackson (1975) for the details.

Thus, in the foregoing integrals, the source terms $\rho$ and $\boldsymbol{j}$ are evaluated not at the present but at an earlier time. It is earlier by the light travel time from the point $\boldsymbol{r}'$ to the position $\boldsymbol{r}$ in question. It is the source term at that time which is affecting the potentials at the present time. The location of a single charge contributing to the potential at the present time is found by running the particle trajectory backward in time until it intersects a sphere of radius $R = c(t-t')$ centered on the observer and expanding at the velocity of light backward in time from the present.

When evaluated for a single moving charge, $\phi$ and $\boldsymbol{A}$ are called the Liénard-Wiechart potentials. In this case, the charge and current densities are given by

$$\rho(\boldsymbol{r},t) = q\,\delta(|\boldsymbol{r}-\boldsymbol{r}_o(t)|), \quad \boldsymbol{j}(\boldsymbol{r},t) = q\boldsymbol{u}(t)\,\delta(|\boldsymbol{r}-\boldsymbol{r}_o(t)|)$$

where $\boldsymbol{r} = \boldsymbol{r}_o(t)$ is the particle trajectory and $\boldsymbol{u} = \dot{\boldsymbol{r}}_o(t)$ is the particle's velocity. The integrals are a bit tricky. Take $\phi$ for example:

$$\phi(\boldsymbol{r},t) = q\int \frac{\delta(|\boldsymbol{r}'-\boldsymbol{r}_o(t')|)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,d^3r' \;.$$

The integration is not as trivial as it looks because $t'$ depends on $r'$ and so the value of $r'$ which makes the $\delta$-function argument zero is not straightforward. Refer to *Radiative Processes*, pp 78-79 for the details.

$$\begin{aligned}\phi(\boldsymbol{r},t) &= \frac{q}{\kappa(t')|\boldsymbol{r}-\boldsymbol{r}_o(t')|} \;, \\ \boldsymbol{A}(\boldsymbol{r},t) &= \frac{q\boldsymbol{u}(t')}{c\kappa(t')|\boldsymbol{r}-\boldsymbol{r}_o(t')|} \;,\end{aligned}$$

where

$$\kappa(t') \equiv 1 - \frac{1}{c}\boldsymbol{n}(t')\cdot\boldsymbol{u}(t')$$

and $\boldsymbol{n}(t')$ is a unit vector pointing towards the observer at time $t'$.

These potentials differ significantly from the static EM potentials, especially at relativistic velocities. In particular, the potentials are very large along the direction of motion if $|\boldsymbol{u}| \sim c$.

The differentiations of the potentials to obtain the fields is lengthy and tedious. Refer to Jackson for the details if you are interested. Define

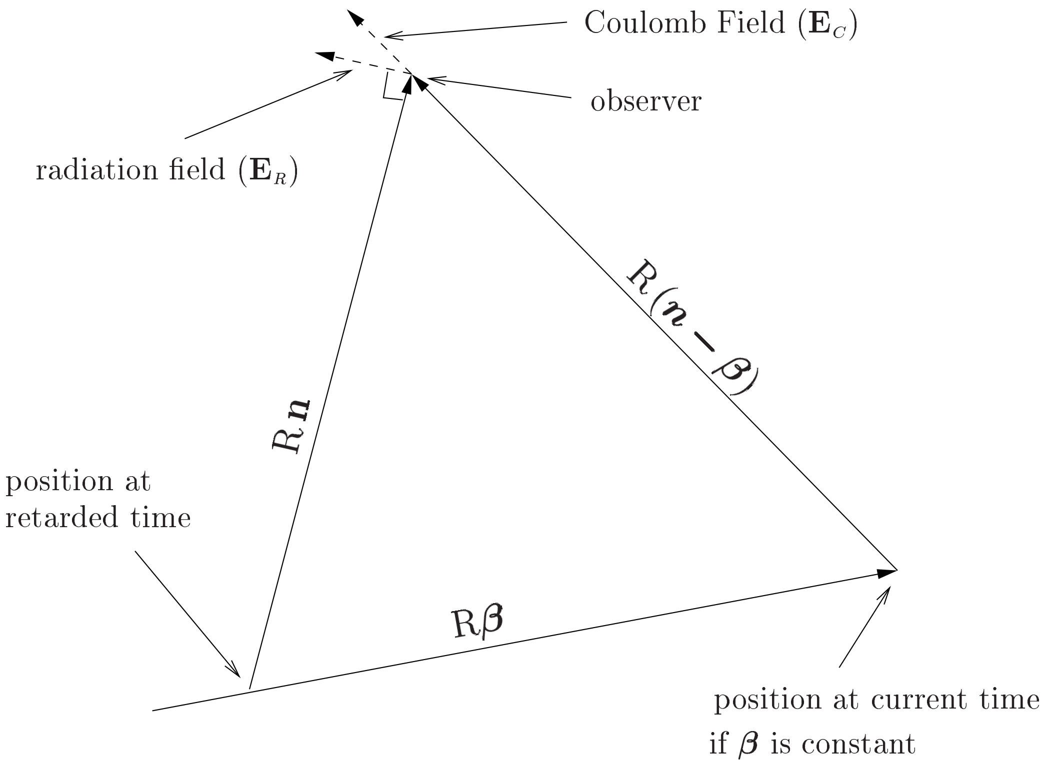
$$\boldsymbol{\beta} \equiv \frac{\boldsymbol{u}(t')}{c} \;, \quad \kappa \equiv 1 - \boldsymbol{n}\cdot\boldsymbol{\beta} \;, \quad R \equiv |\boldsymbol{r}-\boldsymbol{r}_o(t')| \;.$$

Then,

$$
\begin{aligned}
\boldsymbol{E}(\boldsymbol{r},t) &= \frac{q}{\kappa^3 R^2}(\boldsymbol{n}-\boldsymbol{\beta})(1-\beta^2) + \frac{q}{\kappa^3 Rc}[\boldsymbol{n}\times(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}] \;, \\
\boldsymbol{B}(\boldsymbol{r},t) &= \boldsymbol{n}\times\boldsymbol{E}(\boldsymbol{r},t) \;,
\end{aligned}
$$

where all terms on the RHS of the equation for $\boldsymbol{E}$ are *evaluated at the retarded time*! Note that the fields consist of two terms. The first is the "Coulomb" term familiar in electrostatics; it falls off like $R^{-2}$. The second is the "radiation" term which falls off like $R^{-1}$. Clearly, in the latter case, the Poynting flux falls off like $R^{-2}$, allowing energy to escape to infinity.

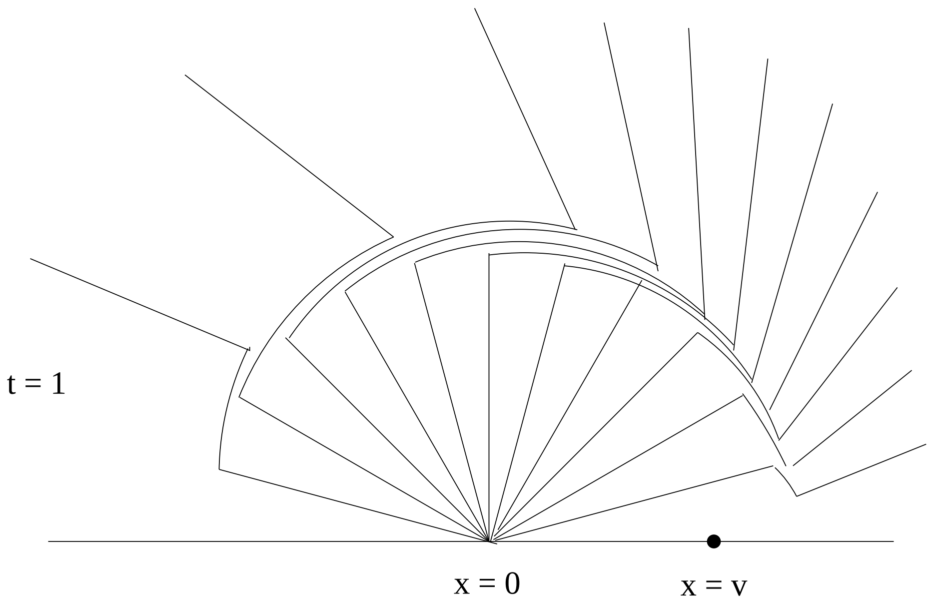
Note too that the $\boldsymbol{E}$-field associated with the Coulomb term points toward $(\boldsymbol{n}-\boldsymbol{\beta})$ which is exactly in the direction pointing *away* from the particle at the *current* time if it moved at a constant speed equal to that at the retarded time.




The diagram shows how the Coulomb field "anticipates" the position of the particle at the current time by performing an extrapolation at constant speed. The vector $R\boldsymbol{\beta}$ is the distance travelled during the interval between the retarded time and the current time. Note that in drawing the above figure, I have assumed for simplicity that the vector $\dot{\boldsymbol{\beta}}$ lies in the plane of the page.

Turning to the radiation field, we see that $\boldsymbol{E}_{\rm rad}$ is normal to $\boldsymbol{n}$ which, of course, is consistent with Maxwell's equations. It is also proportional to the *acceleration* of the charge and this is the key element. Uniformly moving charges don't radiate.

The following diagram shows geometrically how an acceleration can give rise to a transverse field which is stronger than the Coulomb fields. A particle moving along the $x$-axis at constant velocity $v$ is suddenly stopped at $x=0$ at $t=0$. At $t=1$, the field looks like this:




The EM pulse caused by the deceleration is at a distance $c$ and transverse and strong in view of the tightly packed field lines. One can argue on the basis of conservation of field lines at the discontinuity that the field falls off like $R^{-1}$ in the thin layer there.

**Larmor's Formula**

This formula gives the power radiated by a single charge in the case $\beta \ll 1$, and is very useful as we shall see later. Here, we consider only the "far zone"; i.e., large distances where the radiation field dominates. The far zone boundary can be estimated as follows:

$$\frac{E_{\mathrm{R}}}{E_{\mathrm{C}}} \sim \frac{R\dot{u}}{c^2} \sim \frac{Ru}{\tau c^2}$$

where I have associated a characteristic timescale $\tau$ with the acceleration $\dot{u}$ ($\tau \sim u/\dot{u}$). Thus, $E_{\mathrm{R}} \gg E_{\mathrm{C}}$ wherever $R \gg \tau c^2/u$. This inequality defines the far zone; it is a distance at least as great as the distance travelled by light during the acceleration period.

Now, when $\beta \ll 1$, the radiation fields simplify to

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r},t) &= \frac{q}{Rc^2}\,\boldsymbol{n} \times [\boldsymbol{n} \times \dot{\boldsymbol{u}}(t')] = \frac{q}{Rc}\,\boldsymbol{n} \times (\boldsymbol{n} \times \dot{\boldsymbol{\beta}}) \ , \\ \boldsymbol{B}(\boldsymbol{r},t) &= \boldsymbol{n} \times \boldsymbol{E}(\boldsymbol{r},t) \ . \end{aligned}$$

Note that the $\boldsymbol{E}$-vector lies in the plane defined by $\boldsymbol{n}$ and $\dot{\boldsymbol{\beta}}$. Its magnitude is

$$E = B = \frac{q\dot{\beta}}{Rc}\,\sin\theta$$

where $\theta$ is obviously the angle between $\boldsymbol{n}$ and $\dot{\boldsymbol{\beta}}$. Remember that $\dot{\beta}$ is always evaluated at the *retarded* time.

The flux is now

$$|\boldsymbol{S}| = S = \frac{dW}{dAdt} = \frac{c}{4\pi}\,E^2 = \frac{q^2\dot{\beta}^2}{4\pi R^2 c}\,\sin^2\theta \ .$$

To evaluate the total power radiated by the accelerating charge, we have to integrate the flux over the surface of a sphere of radius $R$. In spherical coordinates, the element of surface area is

$$dA = 2\pi R^2 \sin\theta \, d\theta \ .$$

Thus,

$$P = \frac{dW}{dt} = \frac{q^2\dot{\beta}^2}{4\pi R^2 c}\int_0^\pi \sin^2\theta \ 2\pi R^2 \sin\theta \, d\theta = \frac{q^2\dot{\beta}^2}{2c}\int_0^\pi \sin^3\theta \, d\theta \ .$$

But over a sphere, $\langle\sin^2\theta\rangle = 2/3$; therefore,

$$P = \frac{2q^2\dot{\beta}^2}{3c} = \frac{2q^2\dot{u}^2}{3c^3} \ .$$

The power can also be written in terms of the acceleration of the dipole moment: set $\boldsymbol{d} \equiv q\boldsymbol{r}$ from which it follows that $\ddot{\boldsymbol{d}} = q\ddot{\boldsymbol{r}} = q\dot{\boldsymbol{u}}$. Hence,

$$P = \frac{2|\ddot{\boldsymbol{d}}|^2}{3c^3} \ .$$

<u>Notes</u>:

1. The radiation is characterized by a "ring" or "annular" pattern (i.e., it's distributed like $\sin^2\theta$). No radiation is emitted along the direction of the acceleration and the maximum is emitted perpendicular to $\dot{\boldsymbol{u}}$.

2. If the particle accelerated along a line, the radiation will be 100% *linearly* polarized in the plane of $\boldsymbol{n}$ and $\dot{\boldsymbol{u}}$.

3. This is yet another illustration of the fact that most of the radiation in nature is due to oscillating or accelerating electric dipoles.

4. Due to the symmetry of $\boldsymbol{E}$ and $\boldsymbol{B}$ in Maxwell's equations, this equation also applies to accelerating *magnetic* dipoles, a good example of which is a neutron star with a magnetic axis inclined to the rotation axis and precessing about it. The magnetic dipole radiation is very large in this case and the neutron star or pulsar spins down rapidly as a result. Relevant timescales are $10^3$ to $10^6$ years.

## Larmor Radiation Spectrum

We learned earlier that the spectrum of the flux averaged over the signal interval $T$ is

$$\frac{dS}{d\nu} = \frac{dP}{dA\,d\nu} = \frac{dW}{dA\,d\nu\,dt} = \frac{c}{2\pi}\,\frac{|\widehat{E}(\nu)|^2}{T}\ ,$$

allowing for negative frequencies. Therefore, the total energy emitted per unit bandwidth per unit area is

$$\frac{dW}{dA\,d\nu} = \frac{dS}{d\nu}\,T = \frac{c}{2\pi}\,|\widehat{E}(\nu)|^2\ .$$

Suppose now that we consider a time varying or oscillating dipole moment whose magnitude is characterized by $d(t)$. Then, the field transform $\widehat{E}(\nu)$ may be expressed in terms of the transform of $\ddot{d}(t)$, namely, $\widehat{\ddot{d}}(\nu)$:

$$\widehat{E}(\nu) = \frac{\widehat{\ddot{d}}(\nu)}{Rc^2}\,\sin\theta = -\frac{4\pi^2\nu^2}{Rc^2}\,\widehat{d}(\nu)\,\sin\theta\ .$$

Substituting, we now have

$$\frac{dW}{dA\,d\nu} = \frac{c}{2\pi}\,\frac{16\pi^4\nu^4}{R^2c^4}\,\widehat{d}(\nu)^2\,\sin^2\theta = \frac{8\pi^3\nu^4}{R^2c^3}\,|\widehat{d}(\nu)|^2\,\sin^2\theta\ .$$

This is the energy per unit area per unit band width. To get the total energy per unit band width, we again must integrate over the surface of a sphere of radius $R$. This introduces a factor $\frac{8}{3}\pi R^2$ as we saw before. Thus,

$$\frac{dW}{d\nu} = \frac{64\pi^4\nu^4}{3c^3}\,|\widehat{d}(\nu)|^2\ .$$

In *Radiative Processes* (p88), this equation is written as

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}\,|\widehat{d}(\omega)|^2$$

which is easy to obtain from our formula by setting $\omega = 2\pi\nu$ and $\widehat{d}(\nu) = 2\pi\widehat{d}(\omega)$.

There is another useful expression involving the spectrum of the energy emitted during a transient acceleration (e.g., an encounter or pulse). This will be useful later in computing the spectrum of bremsstrahlung or free-free emission:

$$P = \frac{2q^2}{3c^3}\,|\dot{\boldsymbol{u}}|^2 = \frac{2q^2}{3c^3}\,(\ddot{x}^2 + \ddot{y}^2) \quad \text{(motion in a plane defined by } \boldsymbol{u} \text{ and } \dot{\boldsymbol{u}})\ .$$

The total energy emitted during a “pulse” is

$$W = \int_{-\infty}^{\infty} P\,dt = \frac{2q^2}{3c^3}\int_{-\infty}^{\infty} (\ddot{x}^2 + \ddot{y}^2)\,dt\ .$$

Now invoke the power theorem again:

$$\begin{aligned}
\int_{-\infty}^{\infty} \ddot{x}^2(t)\,dt &= \int_{-\infty}^{\infty} |\widehat{\ddot{x}}(\nu)|^2\,d\nu\ ,\\
\int_{-\infty}^{\infty} \ddot{y}^2(t)\,dt &= \int_{-\infty}^{\infty} |\widehat{\ddot{y}}(\nu)|^2\,d\nu\ .
\end{aligned}$$

It follows that the spectrum of the emitted radiation is

$$\frac{dW}{d\nu} = \frac{4q^2}{3c^3}\left(|\widehat{\ddot{x}}(\nu)|^2 + |\widehat{\ddot{y}}(\nu)|^2\right)$$

where I have thrown in a factor 2 to allow for negative frequencies.

## 2.2 Relativistic Treatment

### Relativistic Larmor Formula

For a relativistic particle, we may compute the total radiated power by transforming to a frame in which the particle is essentially at rest (momentarily because it may be accelerating). This is made easy by the fact that the emitted power is invariant under Lorentz transformations. To see this, let

$$P = \frac{dW}{dt} \ \text{(inertial)} \ , \qquad P' = \frac{dW'}{dt'} \ \text{(particle rest frame)} \ .$$

The Doppler formula (p112 of text) is

$$\nu = \nu'\gamma\left(1 + \frac{v}{c}\cos\theta'\right) \quad \text{where} \quad \gamma \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad \text{and} \quad \cos\theta' = \frac{\boldsymbol{n}\cdot\boldsymbol{u}}{|\boldsymbol{u}|} \ .$$

Again, the prime denotes the particle rest frame in which the particle emits symmetrically with a ring pattern. This symmetry allows us to average over angle in a straightforward way to obtain

$$\langle\nu\rangle = \nu'\gamma \quad \text{and so} \quad dW = \gamma\, dW'$$

where the brackets indicate an average over angle weighted by the power in each direction. But also from the Lorentz transformation, $dt = \gamma\, dt'$ which means that both energy and time are NOT Lorentz invariant! Therefore,

$$P = \frac{dW}{dt} = \frac{dW'}{dt'} = P' \ .$$

Recall that the Larmor formula is

$$P = P' = \frac{2q^2}{3c^3}\,|\boldsymbol{a}'|^2 = \frac{2q^2}{3c^3}\,(\dot{u}')^2$$

where $\boldsymbol{a}'$ is the acceleration in the rest frame. We now express $\boldsymbol{a}'$ in terms of $\boldsymbol{a}$ using the Lorentz transformation. First define $a_{\|} = \dot{u}$ (parallel to $\boldsymbol{u}$) and $a_{\perp} = \dot{w}$ (perpendicular to $\boldsymbol{u}$). Then invoke the Lorentz transformation:

$$x = (x' + vt')\gamma \ , \quad y = y' \ , \quad t = \left(t' + \frac{v}{c^2}x'\right)\gamma \ .$$

<u>First, transform $a_{\|}$:</u>

$$u = \frac{dx}{dt} = (u' + v)\gamma\frac{dt'}{dt} = \frac{(u'+v)\gamma}{\left(1 + \frac{u'v}{c^2}\right)\gamma} = \frac{u'+v}{1 + \frac{u'v}{c^2}} \ .$$

Therefore,

$$\begin{aligned}
a_{\|} = \dot{u} \quad &= \quad \frac{\left(1 + \frac{u'v}{c^2}\right)\dot{u}' - (u'+v)\left(\frac{\dot{u}'v}{c^2}\right)}{\left(1 + \frac{u'v}{c^2}\right)^2}\,\frac{dt'}{dt} \\
&= \quad \frac{\left(1 - \frac{v^2}{c^2}\right)\dot{u}'}{\left(1 + \frac{u'v}{c^2}\right)^2\left(1 + \frac{u'v}{c^2}\right)\gamma} \\
&= \quad \frac{a'_{\|}}{\gamma^3\left(1 + \frac{u'v}{c^2}\right)^3} \ .
\end{aligned}$$

But since $u' = 0$ (particle at rest in its own frame),

$$a'_{||} = \gamma^3 a_{||} \ .$$

Next consider $a_\perp$:

$$w = \frac{dy}{dt} = w' \frac{dt'}{dt} = \frac{w'}{\gamma\left(1 + \frac{u'v}{c^2}\right)} \ .$$

Therefore,

$$\begin{aligned} a_\perp = \dot{w} &= \frac{\left(1 + \frac{u'v}{c^2}\right)\dot{w}' - w'\left(\frac{\dot{u}'v}{c^2}\right)}{\left(1 + \frac{u'v}{c^2}\right)^2 \gamma} \frac{dt'}{dt} \\ &= \frac{\left(\dot{w}' + \frac{u'v\dot{w}'}{c^2} - \frac{\dot{u}'w'v}{c^2}\right)}{\left(1 + \frac{u'v}{c^2}\right)^3 \gamma^2} \ . \end{aligned}$$

But again, $u' = w' = 0$ in the rest frame; hence,

$$a'_\perp = \gamma^2 a_\perp \ .$$

Thus, substituting, we finally obtain

$$P = \frac{2q^2\gamma^4}{3c^3}\left(a_\perp^2 + \gamma^2 a_{||}^2\right) \ .$$

One sees that for a relativistic particle a *small* (apparent) acceleration in the inertial (fixed) frame is accompanied by a *large* energy loss! What's happening here is that large accelerations in the rest frame appear small in the observer's frame because to the observer clocks in the source run slower and rulers are shorter. Thus, the $\gamma^4$ factor allows for the time dilation whereas the $\gamma^2$ term allows for the spatial contraction in the direction of motion.

**Lorentz Invariants**

For future reference, let's summarize the Lorentz invariants:

$$P, \ \ \frac{I_\nu}{\nu^3}, \ \ \frac{S_\nu}{\nu^3}, \ \ \frac{j_\nu}{\nu^2}, \ \ \nu\alpha_\nu \ .$$

The invariance of the last four follows from the invariance of the phase space density, $f \equiv dN/dV_{\rm ph}$ (both number and volume are invariant: $dN = dN'$ and $dV_{\rm ph} = dV'_{\rm ph}$). Thus, $h\nu \, f \, 4\pi p^2 dp = U_\nu d\nu = 4\pi(I_\nu/c)d\nu$, from which we conclude that $I_\nu/\nu^3 \propto f = \text{const}$.

Note that the following are *not* invariant:

$$W, \ \ t, \ \ \boldsymbol{E}, \ \ \text{and} \ \ \boldsymbol{B} \ .$$

The energy $W$ is just one element of a 4-vector which changes during a Lorentz transformation because the latter corresponds simply to a rotation of a 4-vector in 4-dimensional space. Also, during a transformation, E-fields can be converted into B-fields and vice-versa.

## Angular Distribution of Emitted Power

Even more interesting is that the angular distribution of the emission changes dramatically for a relativistic particle. See section 4.8 in *Radiative Processes* (pp 138-145) for the details. The main results are summarized in the diagram on the next page. Note the strong beaming effect in the direction of motion. Fig. 2.24 is the most relevant for particles spiraling about magnetic field lines – in that situation, the acceleration is normal to the velocity; i.e., $\boldsymbol{a} \propto (\boldsymbol{v} \times \boldsymbol{B})$.

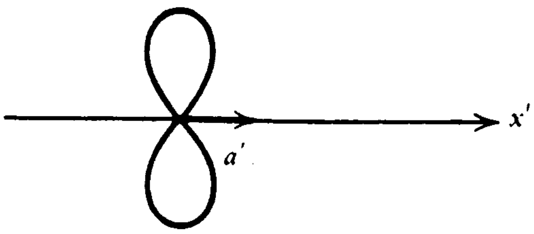



**Figure 2.21** – Annular or ring radiation pattern for particle at rest.

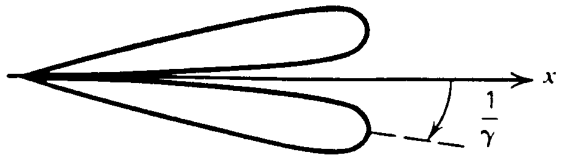



**Figure 2.22** – Angular distribution of radiation emitted by a relativistic particle with an acceleration that is *parallel* to its velocity along the $x$-axis.

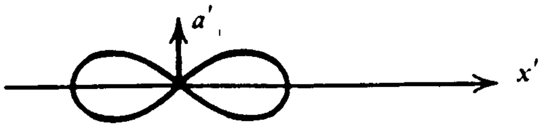



**Figure 2.23** – Same as Fig. 2.21.

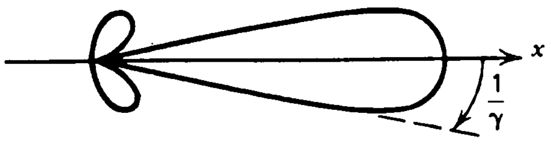



**Figure 2.24** – Angular distribution of radiation by a relativistic particle with an acceleration that is *perpendicular* to its velocity along the $x$-axis.

## 2.3 Particle Ensembles

The preceding formulae apply to a *single* accelerated particle. If there are many radiating particles, then we could, in principle, just sum up the contributions from the many sources. But there is an obvious complication in that the retarded times for each particle are different. Thus, to do this in a rigorous way, we would have to keep track of the phase relations between the various parts of the radiating system.

However, in general, we don't worry about such complexities. We just assume that the particles radiate incoherently and sum up the contributions in a simple-minded way. In addition, we generally assume randomly oriented magnetic fields in which case the polarizations are also incoherent. That is, we do not expect any net polarization. On the other hand, if there are uniform fields over large volumes, we might have a correlation of polarization planes but no phase coherence. This can happen in the case of synchrotron radiation.

Of course, if the particle velocities are much less than $c$, then we don't have to worry about retarded times.

## 2.4 Thomson Scattering

An important application of Larmor's dipole formula is electron scattering of radiation by free (unbound) charges. Electrons play a much bigger role here than ions due to their low masses as we shall see.

If the charge oscillates at nonrelativistic velocities, then we may neglect the magnetic force $\frac{e}{c}\boldsymbol{v} \times \boldsymbol{B}$ relative to the electric one $e\boldsymbol{E}$ since $E_0 = B_0$ for an EM-wave. Also, to ensure that we can ignore the forward momentum carried by radiation, we have to require $h\nu \ll mc^2$. In this regime, electron scattering is called "Thomson scattering". The more general case (high and low $\nu$) is called "Compton scattering" – a noncoherent generalization of Thomson scattering.

Assume a linearly polarized, incident wave:
The force is

$$F = eE_0 \sin\omega_0 t$$

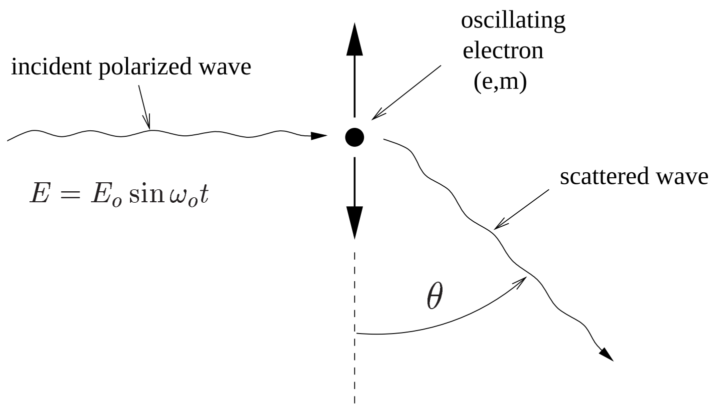
yielding an acceleration

$$\ddot{r} = \frac{e}{m} E_0 \sin\omega_0 t \ .$$

Solving for the displacement, we find

$$r = -\frac{e}{m{\omega_0}^2} E_0 \sin\omega_0 t \ .$$

Note that $r$ and $E$ are in phase for a *negative* charge like the electron.




Hence, the Poynting flux becomes

$$S = \frac{e^2 \dot{u}^2}{4\pi R^2 c^3} \sin^2\theta = \frac{e^4}{4\pi m^2 R^2 c^3} {E_0}^2 \sin^2\omega_0 t \sin^2\theta \ .$$

The time-averaged flux is

$$\langle S \rangle = \frac{e^4 E_0{}^2}{8\pi m^2 R^2 c^3} \sin^2\theta \qquad \text{(frequency independent!)}$$

What is the power per unit solid angle, $dP/d\Omega$?

$$dP = \langle S \rangle \, R^2 \, d\Omega$$

where $R^2 \, d\Omega$ is, of course, an element of area on the surface of a sphere. Thus,

$$\frac{dP}{d\Omega} = \frac{e^4 E_0{}^2}{8\pi m^2 c^3} \sin^2\theta \; .$$

Note the $m^{-2}$ factor which makes ions unimportant as scattering agents. Another point is that the scattering is *nonisotropic*.

The integrated (or total) scattered power is easily obtained by integrating over all solid angles. This introduces a factor $8\pi/3$ as we have seen:

$$P = \frac{e^4 E_0{}^2}{3 m^2 c^3} \; .$$

To get the corresponding scattering cross-section, recall that the flux associated with an incident plane wave is

$$\langle S \rangle_i = \frac{c}{4\pi} \langle E^2 \rangle = \frac{c}{8\pi} E_0{}^2 \; .$$

Then defining

$$\frac{d\sigma}{d\Omega} = \frac{1}{\langle S \rangle_i} \frac{dP}{d\Omega} \qquad \text{(differential cross-section)}$$

and

$$\sigma = \oint \frac{d\sigma}{d\Omega} \, d\Omega = \frac{P}{\langle S \rangle_i} \qquad \text{(total cross-section)} \, ,$$

we have

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{m^2 c^4} \sin^2\theta = r_o^2 \sin^2\theta \; ,$$

and

$$\sigma = \frac{8\pi e^4}{3 m^2 c^4} = \frac{8\pi}{3} r_o^2 \; .$$

In these equations, $r_o \equiv e^2/mc^2$ is the classical radius of the electron. The quantity $\sigma$ is called the Thomson cross-section and has the numerical value

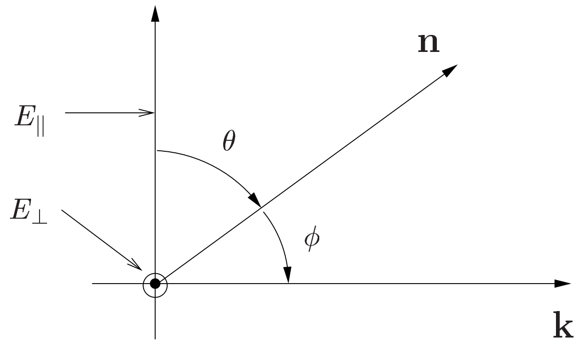
$$\sigma = \sigma_{\mathrm{T}} = 6.65 \times 10^{-25} \, \mathrm{cm}^2 \quad \text{(frequencey independent)} \, .$$

When $h\nu \geq mc^2$, we must use the Compton scattering cross-section which *does* depend on the frequency. In particular, it *decreases* with $\nu$ because the electron becomes less and less efficient as a scatterer due to its own inertia. But even the Compton cross-section is not valid for radiation fields sufficiently intense that the electron moves at relativistic speeds.

**Polarization of Scattered Radiation**

We have already seen that the $\boldsymbol{E}$-vector of the scattered radiation is polarized in a plane containing the acceleration vector and the direction of propagation; i.e., the $\boldsymbol{E}$-vector is in the $(\dot{\boldsymbol{u}}, \boldsymbol{n})$-plane and perpendicular to $\boldsymbol{n}$ itself.

In deriving the cross-section, we have assumed that incident radiation is *polarized*. To get the differential cross-section for unpolarized incident radiation, we work out the separate cross-sections for two orthogonal, linear polarizations and add them (valid if the two are uncorrelated which they must be if the radiation is unpolarized). We can choose orthogonal polarizations in any direction, so choose the first to be in the plane containing $\boldsymbol{n}$ and $\boldsymbol{k}$ where $\boldsymbol{n}$ is the direction of the scattered emission (arbitrary angle) and $\boldsymbol{k}$ is the incident direction. If $\phi$ is the angle between $\boldsymbol{n}$ and $\boldsymbol{k}$, then for this polarization $\theta = \frac{\pi}{2} - \phi$ and the cross-section is




$$\left(\frac{d\sigma}{d\Omega}\right)_{\|} = r_o^2 \cos^2\phi \qquad (\text{for } E_{\|}) \ .$$

For the orthogonal direction, $\theta = \frac{\pi}{2}$ and so

$$\left(\frac{d\sigma}{d\Omega}\right)_{\perp} = r_o^2 \qquad (\text{for } E_{\perp}) \ .$$

The total differential cross-section for the unpolarized beam is just the average of the two:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{2} r_o^2 \, (1 + \cos^2\phi) \ .$$

Then the total cross-section becomes

$$\sigma_{\text{unpol}} = \oint \left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} d\Omega = \frac{1}{2} r_o^2 \, (4\pi + \frac{4\pi}{3}) = \frac{8\pi}{3} r_o^2 = \sigma_{\text{pol}} \ .$$

Note that for this integration $\phi$ is really the *polar* angle, not the azimuthal one, and we have used the fact that the mean of $\cos^2\phi$ over a sphere is exactly $1/3$. (Can you explain why the mean of $\cos^2\phi$ is different from the mean of $\sin^2\phi$ which is $2/3$ as we saw earlier?)

We see here that the cross-sections for polarized and unpolarized radiation are the *same* which shouldn't be a surprise because our unpolarized beam is just the superposition of two orthogonal polarized beams, each of which see the same cross-section. This raises the following question: what if the incident beam were circularly polarized or elliptically polarized? Would the cross-sections be the same?

The degree of polarization is easy to obtain when one realizes that the polarized intensities in the plane and perpendicular to the plane of scattering are in the ratio $\cos^2\phi : 1$. Thus,

$$\Pi = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} = \frac{1 - \cos^2\phi}{1 + \cos^2\phi} \qquad (\text{unpolarized incident beam}) \ .$$

We have the interesting result that even an unpolarized beam produces a scattered wave with some polarization, the degree of which depends on the viewing angle. At 90° to the incident wave, we see 100% polarization and the explanation for this should be obvious.

We can guess what happens to the polarization of a circularly polarized incident beam. Looking along the beam, one would see 100% circular polarization. At right-angles to the beam, on the other hand, the polarization would be 100% *linear*.

## 2.5 Radiation from Oscillating Bound Charges

I want to consider now radiation from oscillating *bound* charges such as an electron in an atom. And we want to make a *classical* analysis. Such an approach would appear doomed to failure because this is inherently a quantum mechanical problem. But, believe it or not, the harmonically bound oscillator problem was tackled initially as a classical problem using a "toy model" for the atom: one imagines that the electron is a harmonic oscillator whose natural frequency $\omega_o$ is determined by the electron-nucleus interaction. If one assumes that this simple system radiates according to Maxwell's equations, the result is a damped harmonic oscillator whose Fourier transform yields a broadened profile. This is just an academic application of Maxwell's equations; there is really no physics here but the approach has had some historical successes which are worth noting. In particular, the classical theory provides

- the correct profile (and width) for collision broadening and radiation damping;
- the correct Rayleigh scattering formula ($\lambda^{-4}$ law);
- an explanation as to why the index of refraction $n$ is larger than unity in water but less than unity in an ionized plasma;
- a prediction for the approximate absorption coefficients (oscillator strengths) for strong bound-bound transitions;
- the origin of the term "oscillator strength".

### Radiation Reaction

This is a useful (but classical only) concept to introduce before we discuss harmonically bound particles because it deals with the energy loss experienced by all radiating systems. We take a slightly different approach than the text.

First of all, we restrict our attention to an oscillating system in which the energy lost during one cycle (or period) is small. Then the motion can be closely approximated by

$$x = x_o \cos \omega_o t$$

for many cycles. The *radiation reaction* is the force that must act back on the particle, slowing it down and depriving it of its energy. Thus, the work done by the force on the particle is *negative*:

$$\langle \boldsymbol{F}_{\text{rad}} \cdot \boldsymbol{u} \rangle = -\left\langle \frac{2e^2 |\dot{\boldsymbol{u}}|^2}{3c^3} \right\rangle \quad \text{(average over one cycle).}$$

Now, since the oscillation is envisaged to be in one coordinate only, $\boldsymbol{F}$ must act along this dimension. Also, if $\boldsymbol{F} \cdot \boldsymbol{u}$ is to be *negative*, then $\boldsymbol{F}_{\text{rad}}$ and $\boldsymbol{u}$ must have opposite senses, so that

$$\boldsymbol{F}_{\text{rad}} \propto -\boldsymbol{u} \quad \text{or} \quad \boldsymbol{F}_{\text{rad}} = -\alpha \boldsymbol{u} \quad \text{(say)}$$

on average, at least. To find the constant of proportionality $\alpha$, average over one cycle as follows:

$$\int_{\text{cycle}} (\boldsymbol{F}_{\text{rad}} \cdot \boldsymbol{u})\, dt = -\int_{\text{cycle}} \frac{2e^2|\dot{\boldsymbol{u}}|^2}{3c^3}\, dt$$

or

$$\alpha \int_{\text{cycle}} u^2\, dt = \frac{2e^2}{3c^3} \int_{\text{cycle}} \dot{u}^2\, dt \ .$$

Note that this equation equates the decline in the kinetic energy during one period with the energy radiated away, confirming our guess that $\boldsymbol{F}_{\text{rad}} \propto -\boldsymbol{u}$. Substituting, we have

$$\alpha x_o^2 \omega_o^2 \langle \sin^2 \omega_o t \rangle = \frac{2e^2}{3c^3}\, x_o^2 \omega_o^4\, \langle \cos^2 \omega_o t \rangle$$

which yields

$$\alpha = \frac{2e^2}{3c^3}\, \omega_o^2 \ .$$

Therefore,

$$\boldsymbol{F}_{\text{rad}} = -\frac{2e^2}{3c^3}\, \omega_o^2 \boldsymbol{u} \qquad (\omega_o^2 \boldsymbol{u} = \ddot{\boldsymbol{u}})\, .$$

Now define

$$\tau \equiv \frac{2e^2}{3mc^3} \ .$$

This has dimensions of *time* and is roughly equivalent to the light crossing time of the electron radius:

$$\tau_e \equiv \frac{r_o}{c} = \frac{e^2}{mc^3} = \frac{3}{2}\tau \simeq 10^{-23}\, \text{s}\, .$$

Thus,

$$\boldsymbol{F}_{\text{rad}} = -m\tau\omega_o^2 \boldsymbol{u}$$

which you may recognize as a "viscosity-like" force, proportional to $-\boldsymbol{u}$.

If our approximation is to be valid (i.e., if the energy lost per cycle is to be negligible), then the timescale for the kinetic energy loss, $T_{\text{E}}$, must greatly exceed the period $P = 1/\omega_o$; that is,

$$T_{\text{E}} \simeq \frac{m\langle u^2 \rangle}{m\tau \langle \dot{u}^2 \rangle} = \frac{\omega_o^2}{\tau \omega_o^4} = \frac{P^2}{\tau}$$

or

$$\frac{T_{\text{E}}}{P} = \frac{P}{\tau} \ .$$

Therefore,

$$\frac{T_{\text{E}}}{P} \gg 1 \quad \text{implies} \quad \frac{P}{\tau} \gg 1 \quad \text{or} \quad \omega_o \tau \ll 1 \ .$$

Thus, the period of oscillation must greatly exceed the light-crossing time for a distance $r_o$. This is generally true. For example, in the visible, $\omega_o \sim 10^{15}\, \text{s}^{-1}$ and $\tau \sim 10^{-23}$ s from which it follows that $\omega_o \tau \simeq 10^{-8}$.

This concept of radiation reaction has no real practical application that I know of, except for the one on the following pages. I think it is a *classical construct* introduced to compensate for the inability of classical EM theory to deal properly with momentum conservation, something which is done properly in quantum mechanics where the momentum of the photon is explicitly allowed for.

**Undriven, Harmonically Bound Oscillator**

I present here a very simple model for the decay of an excited atom. Again, we deviate slightly from the text (Rybicki & Lightman) in the treatment of the power spectrum. Let's begin by writing the equation of motion for a classical harmonic oscillator:

$$\boldsymbol{F} = -k\boldsymbol{r} = -m\omega_o^2\boldsymbol{r} \; .$$

With $\boldsymbol{F}_{\rm rad} = -m\tau\omega_o^2\boldsymbol{u}$, we get

$$m\ddot{x} = -m\omega_o^2 x - m\omega_o^2\tau\dot{x}$$

where the second term on the RHS is very much smaller than the first because $\tau\dot{x} \sim \tau\omega_o x \sim 10^{-8}x$ as shown on the preceding page. Therefore, this is an example of very weak damping which corresponds to a very narrow line. A quantum mechanical treatment would result in even weaker damping since bound electrons can't really radiate as efficiently as Maxwell's equations predict. Thus, we have

$$\ddot{x} + \omega_o^2\tau\dot{x} + \omega_o^2 x = 0 \; .$$

This equation admits solutions of the form $e^{\alpha t}$. Substitute to obtain

$$\alpha^2 + \omega_o^2\tau\alpha + \omega_o^2 = 0$$

which has the solution

$$\alpha = -\Gamma \pm \Gamma\left(1 - \frac{4}{\omega_o^2\tau^2}\right)^{1/2} \; ,$$

where $\Gamma \equiv \omega_o^2\tau/2 \ll \omega_o$ because $\omega_o\tau \ll 1$ as shown earlier. Since $\omega_o\tau \ll 1$, the roots here are *complex*:

$$\alpha = -\Gamma \pm i\omega_o \; .$$

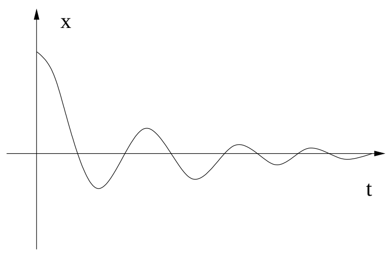
We want $x(t)$ to be *real* and so let's arbitrarily choose the following initial conditions:

$$\left.\begin{array}{rcl} x(0) & = & x_o \\ \dot{x}(0) & = & 0 \end{array}\right\} \quad t = 0 \; .$$

Our solution is a *damped* harmonic oscillator:

$$x(t) = x_o\,e^{-\Gamma t}\cos\omega_o t$$

where $\Gamma$ is the damping constant.




It is easy to demonstrate that the radiation spectrum is peaked at $\omega_o$ with a half width at half maximum governed by $\Gamma$ which is much smaller than $\omega_o$ as shown above. Let's use the angular frequency formulation for $dW/d\omega$, the total energy emitted per unit bandwidth:

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}\,|\widehat{d}(\omega)|^2 \qquad \text{where} \qquad \widehat{d}(\omega) = e\widehat{x}(\omega) \; .$$

Set

$$x(t) = \frac{1}{2}x_o\left(e^{-\Gamma t + i\omega_o t} + e^{-\Gamma t - i\omega_o t}\right) \; .$$

The Fourier transform is then

$$\begin{aligned}\widehat{x}(\omega) &= \frac{x_o}{4\pi}\int_0^\infty \left(\mathrm{e}^{-\Gamma t - i(\omega-\omega_o)t} + \mathrm{e}^{-\Gamma t - i(\omega+\omega_o)t}\right) dt \\ &= \frac{x_o}{4\pi}\left[\frac{1}{\Gamma + i(\omega-\omega_o)} + \frac{1}{\Gamma + i(\omega+\omega_o)}\right] .\end{aligned}$$

Note that we carry out the integral here from 0 to $\infty$ because $x(t)$ is defined only in the interval $0 \le t \le \infty$. Now, let's retain only the positive frequency contribution since our formula for $dW/d\omega$ already incorporates a factor 2 to allow for negative contributions. Therefore,

$$\widehat{x}(\omega) = \frac{x_o}{4\pi}\left[\frac{1}{\Gamma + i(\omega-\omega_o)}\right]$$

or

$$|\widehat{x}(\omega)|^2 = \left(\frac{x_o}{4\pi}\right)^2 \left[\frac{1}{(\omega-\omega_o)^2 + \Gamma^2}\right] .$$

Thus,

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}\left(\frac{ex_o}{4\pi}\right)^2 \left[\frac{1}{(\omega-\omega_o)^2 + \Gamma^2}\right] .$$

This spectrum is small everywhere except near $\omega_o$. Therefore, set $\omega = \omega_o$ in the numerator and recall that

$$\Gamma = \frac{e^2\omega_o^2}{3mc^3} \quad \text{and} \quad k = m\omega_o^2 .$$

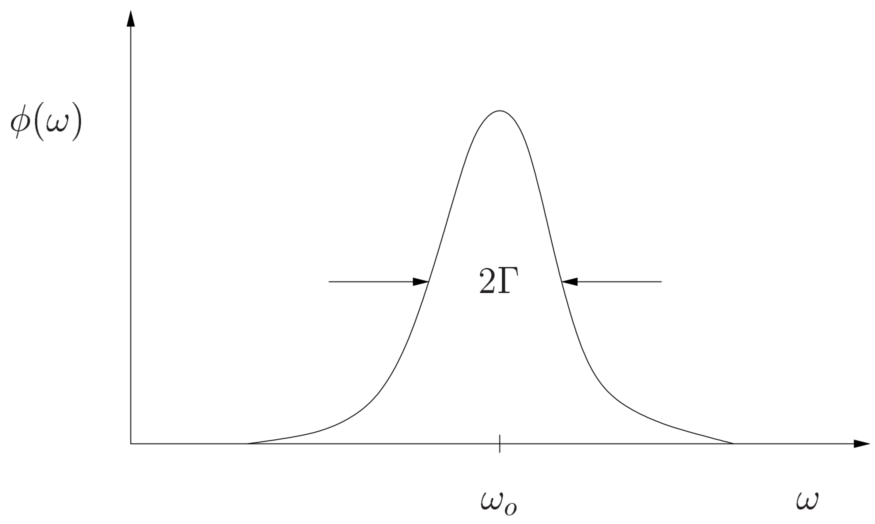
Then $dW/d\omega$ can be rewritten as

$$\frac{dW}{d\omega} = \frac{kx_o^2}{2\pi}\left[\frac{\Gamma}{(\omega-\omega_o)^2 + \Gamma^2}\right] \equiv \frac{kx_o^2}{2}\,\phi(\omega) ,$$

where $\phi(\omega)$ is called the "Lorentz profile". The integral of $\phi(\omega)$ over all frequency is unity and so

$$\int \frac{dW}{d\omega}\, d\omega = \frac{1}{2}kx_o^2 = \text{ total potential energy stored originally in the oscillator.}$$

The width (FWHM) of the Lorentz profile is $2\Gamma$:



It is easy to estimate a nominal value for the lifetime of the upper state:

$$\Gamma = \frac{1}{2}\omega_o^2\tau \sim 10^{31}\,\mathrm{s}^{-2} \times 10^{-23}\,\mathrm{s} \sim 10^{8}\,\mathrm{Hz}$$

if we take $\omega_o = 10^{15.5}\,\mathrm{rad\,s}^{-1}$, an optical frequency. Thus, the decay time of the damped harmonic oscillator is

$$\Delta t \sim \frac{1}{\Gamma} \sim 10^{-8}\,\mathrm{s}$$

and it gets shorter for higher frequencies or higher energy states.

We are seeing here (again) an expression of the uncertainty principle: $\Delta\omega\,\Delta t \sim 1$!. What we have is the only (?) classical model of a spectral line. One can identify $1/\Gamma$ with the lifetime of the upper state of the transition responsible for the line. That is, $\Gamma \sim A_{21}$, the Einstein coefficient.

In fact, instead of expressing transition probabilities in terms of Einstein coefficients, most modern quantum mechanical results are expressed in terms of "oscillator strengths" or "classical damping widths". Strong lines have oscillator strengths approaching *unity* whereas weak lines may have strengths of only $10^{-4} - 10^{-3}$.

The classical intrinsic line width can be easily found from $\Gamma$:

$$\Delta\lambda = 2\pi c\,\frac{\Delta\omega}{\omega_o^2} = 2\pi c\,\frac{2e^2}{3mc^3} = 2\pi c\,\tau\ .$$

Thus,

$$\Delta\lambda = 2\pi c\,\frac{2r_o}{3c} = \frac{4\pi}{3}r_o = 1.2\times 10^{-4}\,\text{Å}$$

where $1\text{Å} \equiv 10^{-8}\,\mathrm{cm}$. This is actually an *over*-estimate for most atomic transitions because they have oscillator strengths less than unity.

There are, of course, many sources of line broadening, but this is the minimum or *intrinsic* broadening associated with the finite lifetime of the excited state. I find it ironic that, in this classical view, the finite lifetime is due to the fictitious *radiation reaction.* We know, of course, that the bound electrons are not radiating even though they are being constantly accelerated!

## Driven Harmonically Bound Oscillator

Now consider *forced* oscillations due to an incident beam of radiation. We can compute the scattered radiation from this beam by evaluating the acceleration from the equation of motion:

$$m\ddot{x} = -m\omega_o^2 x - m\omega_o^2\tau\dot{x} + eE_0\,e^{i\omega t}\ .$$

This leads directly to the total power radiated as a function of the driving frequency (this looks like a spectrum but it is not):

$$P(\omega) = \frac{e^4 {E_0}^2}{3m^2c^3}\left[\frac{\omega^4}{(\omega^2-\omega_o^2)^2 + (\omega_o^3\tau)^2}\right] \quad \text{(see text, p100)}\ .$$

The cross-section for scattering can be found as before ($\sigma = P/\langle S\rangle$):

$$\sigma(\omega) = \frac{\omega^4\sigma_{\mathrm{T}}}{(\omega^2-\omega_o^2)^2 + (\omega_o^3\tau)^2}$$

where $\sigma_{\mathrm{T}} \equiv$ Thomson scattering coefficient. Consider the following three cases:

(a) $\omega \gg \omega_o: \quad \sigma(\omega) \to \sigma_{\mathrm{T}}$ (binding unimportant),

(b) $\omega \ll \omega_o: \quad \sigma(\omega) \to \sigma_{\mathrm{T}} \left(\frac{\omega}{\omega_o}\right)^4$ (binding dominant).

In this case (the low frequency limit), there are no inertial effects because the electric vector is changing so slowly. This means that the displacement is proportional to $E$ (i.e., $x \propto eE$) and, therefore, the acceleration goes like $\omega^2$ and the power radiated like $\omega^4$. This is "Rayleigh scattering" which is responsible for the blue sky and red sunsets!

The foregoing simple analysis reveals why the dielectric constant and the index of refraction for an ionized plasma differ from those for a neutral gas or fluid. In the first case, the free electrons move in phase with the electric vector ($\boldsymbol{r} \propto -e\boldsymbol{E}/\omega^2$) resulting in $\epsilon < 1$ and $n < 1$ (see the discussion of Thomson scattering on p36). But in the second case, the bound electrons are 180° out of phase ($\boldsymbol{r} \propto e\boldsymbol{E}$) resulting in a different sign for the induced dipole moment $\boldsymbol{P} = (\epsilon - 1)\boldsymbol{E}/4\pi$ and an $\epsilon$ and $n$ which exceed unity.

(c) $\omega \sim \omega_o$:

This case is dominated by the closeness of $(\omega^2 - \omega_o^2)$ to zero. Replace $(\omega^2 - \omega_o^2)^2$ by $(2\omega_o)^2(\omega - \omega_o)^2$ and $\omega$ by $\omega_o$ everywhere else. Then,

$$\sigma(\omega) = \frac{\sigma_{\mathrm{T}}}{2\tau}\left[\frac{\Gamma}{(\omega - \omega_o)^2 + \Gamma^2}\right] \qquad (\Gamma = \frac{1}{2}\omega_o^2\tau)$$

which is the Lorentz profile! Thus, in the neighborhood of a resonance, the shape of the cross-section is the same as the emission from free (undriven) oscillations. The total cross-section for our oscillator is

$$\int_0^\infty \sigma(\nu)\, d\nu = \frac{1}{2\pi}\int_0^\infty \sigma(\omega)\, d\omega = \frac{\pi e^2}{mc} = \text{one oscillator strength.}$$

Note that for atomic absorption lines, it is the *resonance* case which is of interest. That is, we want the absorption coefficient at one of the intrinsic frequencies of the atom. The total absorption coefficient across the line is normally measured in units of $\pi e^2/mc$ and is called the "f-value" or the "oscillator strength" of the transition. Thus, in general, for atomic transitions

$$\int_0^\infty \sigma(\nu)\, d\nu = f_{nn'}\,\frac{\pi e^2}{mc}$$

where the f-value lies between 0 and 1 for essentially all absorption lines.