RADIATION PROCESSES
AST1440F

I. INTRODUCTION

1. Review of Basic Concepts

1.1 Overview of Astrophysical Radiation Mechanisms

As observers, we are interested in mechanisms which are able to generate EM waves that can propagate energy to infinity. Charged particles that are stationary or moving with uniform velocity have EM-fields associated with them. But these fields fall off like r^{-2} (as we shall see later) and so the energy flux (proportional to $\boldsymbol{E} \times \boldsymbol{B}$) falls off like r^{-4}. The result: no energy is radiated away.

In astrophysical environments, there are two basic mechanisms which are accompanied by radiation losses:

1. Charged particles accelerated by

   (a) the Coulomb fields of other charged particles $\Longrightarrow$ *bremsstrahlung* or *free-free emission*,
   (b) magnetic fields $\Longrightarrow$ *synchrotron emission*,
   (c) the oscillating electric fields of EM-waves $\Longrightarrow$ *Compton scattering* which includes *Thomson* and *Inverse Compton scattering.*

   The radiation in all these cases is *continuous* and can be described for the most part by classical radiation theory which is based on Maxwell's equations. Also, the random nature of the processes makes for non-coherent radiation fields.

2. Atomic and molecular transitions which yield both line and continuum emission. Again, due to the random nature of the radiative transitions, there is very little coherence in phase or polarization.

The lack of coherence in astrophysical sources means that we can use the single fluid description of radiation. Some basic atomic (quantum) physics will be needed but we can safely ignore quantum field theory.

This course is devoted to the study of these basic radiation mechanisms (both the theory and the practical applications) and a few related phenomena such as the effects of an ionized plasma on the propagation of EM-waves. Also, in the last section, we shall examine two examples of how radiation can affect the flow of gas by providing channels for cooling or absorption of momentum. The focus will be on the *physics* of the mechanisms which is more or less general and not on *radiative transfer* which is much more environment dependent and specialized for different cases (e.g., stellar interiors, atmospheres, gaseous nebulae, accretion disks, quasars).

You may have noticed that in my list of basic radiation mechanisms there is a glaring omission; namely, *nuclear transitions.* Obviously, this mechanism is of fundamental importance in astrophysics – it is the power source for all stars. But this topic lies beyond our terms of reference; it belongs more rightly in a stellar structure course. As a weak excuse for not dealing with this topic here, I might point out that, as astronomers, we never view directly radiation generated as γ-rays at tens of millions of °K in the deep interiors of stars. The radiation we detect by eye and instrument is always filtered by much cooler matter and comes to us by one or both of the two mechanisms mentioned above; these we will now begin to explore in some detail.

Before doing so, let me point out that most radiation mechanisms can be classified as either "classical" (describable completely by Maxwell's equations) or "atomic" (describable only by quantum mechanics). Examples of the first category are synchrotron radiation and Compton scattering. Examples of the second are the line emission and recombination continuum from a gaseous nebula. Thermal bremsstrahlung, on the other hand, falls neatly in between. The low frequency, low temperature part of the free-free spectrum can be derived classically but the rest (due to close electron-ion encounters) requires a quantum mechanical treatment.

1.2 Electromagnetic Waves

First, consider Maxwell's equations in a vacuum (i.e., the charge and current densities, ρ and $\boldsymbol{j}$, are zero and, in addition, the dielectric constant, ϵ, and the magnetic permeability, μ, are both unity):

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=0\ , \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0\ ,$$
$$\boldsymbol{\nabla}\times\boldsymbol{E}=-\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}\ , \qquad \boldsymbol{\nabla}\times\boldsymbol{B}=\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}\ ,$$

where I am employing Gaussian units (esu + emu). The wave equation follows immediately:

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E})=-\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\boldsymbol{B})=-\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}\ .$$

Now use the vector identity

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{E})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E})-\nabla^2\boldsymbol{E}\ .$$

Invoke $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$ to obtain the wave equation:

$$\nabla^2\boldsymbol{E}=\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}\ .$$

Note that due to the inherent symmetry of Maxwell's equations (they are invariant to the interchanges $\boldsymbol{E}\rightarrow\boldsymbol{B}$ and $\boldsymbol{B}\rightarrow-\boldsymbol{E}$) $\boldsymbol{E}$ and $\boldsymbol{B}$ obey the same wave equation. The solutions for $\boldsymbol{E}$ and $\boldsymbol{B}$ have the form

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{a_1}E_0\,\mathrm{e}^{i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})}\ ,$$
$$\boldsymbol{B}(\boldsymbol{r},t)=\boldsymbol{a_2}B_0\,\mathrm{e}^{i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})}\ ,$$

where $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ are unit vectors specifying the directions of $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively. Note that $\boldsymbol{k}$ is the wave vector: $|\boldsymbol{k}|=2\pi/\lambda=2\pi\nu/c=\omega/c$ which applies to a vacuum or any region where the index of refraction is unity (i.e., the phase velocity $v_p=c$).

These solutions represent *plane waves* traveling in the $\boldsymbol{k}$ direction. Note (i) the maximum phase change with r occurs when $\boldsymbol{k}$ and $\boldsymbol{r}$ are aligned, and (ii) no spatial phase change occurs when $\boldsymbol{k}$ and $\boldsymbol{r}$ are perpendicular to one another.

We can now substitute these solutions into Maxwell's equations and derive some basic identities. We will use the well known basic vector differentiation formulae:

$$\begin{aligned}\boldsymbol{\nabla}\cdot(f\boldsymbol{a}) &= f(\boldsymbol{\nabla}\cdot\boldsymbol{a})+(\boldsymbol{\nabla}f)\cdot\boldsymbol{a}\\ \boldsymbol{\nabla}\times(f\boldsymbol{a}) &= f(\boldsymbol{\nabla}\times\boldsymbol{a})+(\boldsymbol{\nabla}f)\times\boldsymbol{a}\ .\end{aligned}$$

In our application, $\boldsymbol{a}$ is one of the unit (constant) vectors and, therefore, only the second term survives. Thus,

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=\boldsymbol{\nabla}\left(E_0\,\mathrm{e}^{i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})}\right)\cdot\boldsymbol{a_1}=-iE_0(\boldsymbol{k}\cdot\boldsymbol{a_1})\,\mathrm{e}^{i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})}\ .$$

But $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$; therefore, $\boldsymbol{k}\cdot\boldsymbol{a_1}=0$ and, by symmetry, $\boldsymbol{k}\cdot\boldsymbol{a_2}=0$ also.

Similarly,

$$\boldsymbol{\nabla}\times\boldsymbol{E}=\boldsymbol{\nabla}\left(E_0\mathrm{e}^{i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})}\right)\times\boldsymbol{a_1}=-iE_0(\boldsymbol{k}\times\boldsymbol{a_1})\,\mathrm{e}^{i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})}\ .$$

Also,

$$-\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{i\omega}{c} B_0 \boldsymbol{a_2}\, \mathrm{e}^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \ .$$

Equating, we find

$$E_0(\boldsymbol{k}\times\boldsymbol{a_1}) = \frac{\omega}{c} B_0 \boldsymbol{a_2} \ .$$

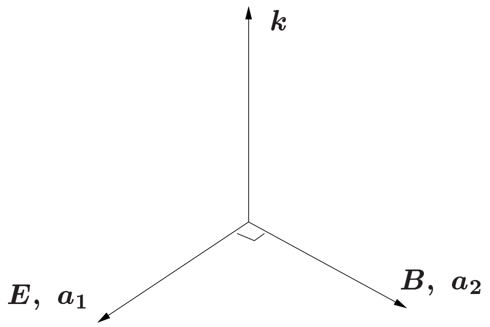
and, again by symmetry,

$$B_0(\boldsymbol{k}\times\boldsymbol{a_2}) = -\frac{\omega}{c} E_0 \boldsymbol{a_1} \ .$$

We can now make the following conclusions:

(a) $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ are *normal* to $\boldsymbol{k}$, the direction of propagation;

(b) $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ are normal to each other as well as to $\boldsymbol{k}$; that is, the three vectors form a mutually orthogonal triad; more specifically, $\boldsymbol{E}$, $\boldsymbol{B}$, and $\boldsymbol{k}$ form a *right-handed* triad as shown in the figure.



(c) $E_0 = \omega B_0/kc$ and $B_0 = \omega E_0/kc$, from which we can conclude that

$$E_0 = \left(\frac{\omega}{kc}\right)^2 E_0 \quad \text{or} \quad \omega^2 = c^2k^2 \quad\Longrightarrow\quad k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \quad\Longrightarrow\quad \lambda\nu = c \ .$$

Thus, the phase velocity of EM-waves in a vacuum is c, a well-known result.

(d) Result (c) implies that $E_0 = B_0$ in Gaussian units.

Poynting's Energy Conservation Theorem

Let's temporarily assume that there is a nonzero current density present so that we can derive a more general result. But assume that the medium is neither dielectric nor permeable (i.e., $\epsilon = \mu = 1$ or $\boldsymbol{D} = \boldsymbol{E}$ and $\boldsymbol{B} = \boldsymbol{H}$). Then, one of Maxwell's equations becomes

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \ .$$

Form the dot-product with $\boldsymbol{E}$:

$$\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) = \frac{4\pi}{c}\boldsymbol{j}\cdot\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}\cdot\boldsymbol{E} \ .$$

Now use the identity

$$\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) - \boldsymbol{\nabla}\cdot(\boldsymbol{E}\times\boldsymbol{B})$$

to obtain

$$-\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{B} - \boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \frac{4\pi}{c}\boldsymbol{j} \cdot \boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \cdot \boldsymbol{E}$$

where $\boldsymbol{\nabla} \times \boldsymbol{E}$ has been replaced by $-\partial \boldsymbol{B}/c\partial t$. Transpose and reorganize to find

$$\frac{\partial}{\partial t}\left(\frac{E^2}{8\pi} + \frac{B^2}{8\pi}\right) = -\boldsymbol{j} \cdot \boldsymbol{E} - \frac{c}{4\pi}\boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{B}) \ .$$

Now introduce the Poynting vector $\boldsymbol{S}$:

$$\boldsymbol{S} \equiv \frac{c}{4\pi}(\boldsymbol{E} \times \boldsymbol{B}) \ .$$

Integrate over an arbitrary volume and apply the divergence theorem:

$$\frac{\partial}{\partial t}\int_V \left(\frac{E^2}{8\pi} + \frac{B^2}{8\pi}\right) dV = -\int_V (\boldsymbol{j} \cdot \boldsymbol{E})\, dV - \int_\Sigma \boldsymbol{S} \cdot d\boldsymbol{A}$$

where Σ is the surface enclosing the volume V. We have here an obvious energy conservation statement:

$$\begin{bmatrix} \text{time rate of change} \\ \text{of EM energy stored} \\ \text{in the fields} \end{bmatrix} = - \begin{bmatrix} \text{mechanical work} \\ \text{done by fields} \\ \text{on charges} \end{bmatrix} - \begin{bmatrix} \text{energy lost} \\ \text{through surface} \\ \text{by radiation} \end{bmatrix} .$$

It's clear that $\boldsymbol{S}$ is the energy flux (flow per unit area) and that $E^2/8\pi$ and $B^2/8\pi$ are energy densities:

$$U_{\mathrm{E}} = \frac{E^2}{8\pi} \quad \left(= \frac{\epsilon E^2}{8\pi}\right) \ , \quad U_{\mathrm{B}} = \frac{B^2}{8\pi} \quad \left(= \frac{B^2}{8\pi\mu}\right) \ .$$

In general, these quantities are variable since $\boldsymbol{E}$ and $\boldsymbol{B}$ are variable. We have seen that for a plane wave $\boldsymbol{E}$ and $\boldsymbol{B}$ are orthogonal and, on average, equal in strength ($E_0 = B_0$). Therefore,

$$\langle S \rangle = \frac{c}{4\pi}\langle |\boldsymbol{E} \times \boldsymbol{B}| \rangle = \frac{c}{4\pi}\langle \boldsymbol{E} \cdot \boldsymbol{E} \rangle = \frac{c}{4\pi}\langle E^2 \rangle$$

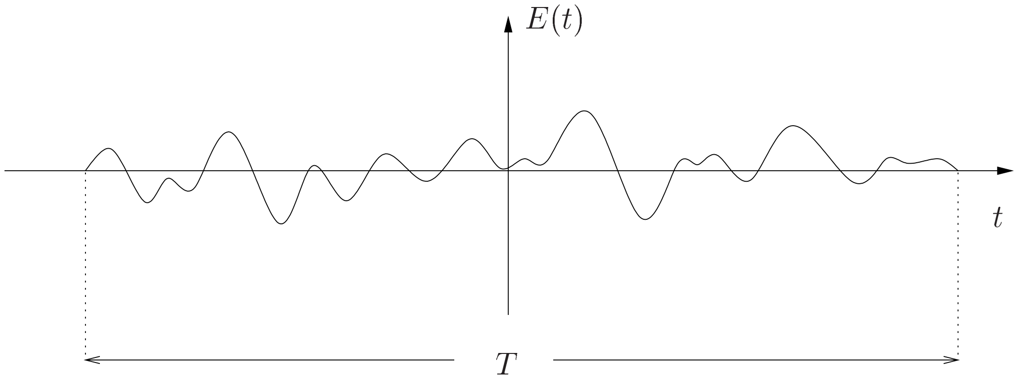
or

$$\langle S \rangle = 2c\langle U_{\mathrm{E}} \rangle = c(\langle U_{\mathrm{E}} \rangle + \langle U_{\mathrm{B}} \rangle) \equiv cU_{\mathrm{rad}}$$

where U_{rad} is the energy density of the radiation field.

1.3 Power Spectra

Consider a wave train or the "pulse" of a randomly varying signal:



We can find the amplitude of this signal at a given frequency by calculating the Fourier transform (cf. R. N. Bracewell, *The Fourier Transform and Its Applications*):

$$\widehat{E}(\nu) = \int_{-\infty}^{\infty} E(t)\, \mathrm{e}^{-2\pi i \nu t}\, dt \ .$$

The inverse of this is

$$E(t) = \int_{-\infty}^{\infty} \widehat{E}(\nu)\, \mathrm{e}^{2\pi i \nu t}\, d\nu \ .$$

Note that formally the frequency spectrum includes *negative* frequencies as well as positive ones but since $E(t)$ is *real*, we can write

$$\widehat{E}(-\nu) = \widehat{E}^*(\nu)$$

where '*' denotes the complex conjugate. Thus, we don't have to use negative frequencies if we don't want to.

Now invoke Parseval's Theorem (Bracewell, p122):

$$\int_{-\infty}^{\infty} E^2(t)\, dt = \int_{-\infty}^{\infty} \widehat{E}(\nu)\widehat{E}^*(\nu)\, d\nu = \int_{-\infty}^{\infty} |\widehat{E}(\nu)|^2\, d\nu \ .$$

With this background, let's proceed to analyze the energy distribution of the pulse. The energy per unit area per unit time is (by Poynting's Theorem)

$$S = \frac{dW}{dt dA} = \frac{c}{4\pi} E^2(t) \ .$$

The *total* energy per unit area in the pulse is therefore

$$\frac{dW}{dA} = \frac{c}{4\pi} \int_{-\infty}^{\infty} E^2(t)\, dt = \frac{c}{4\pi} \int_{-\infty}^{\infty} |\widehat{E}(\nu)|^2\, d\nu = \frac{c}{2\pi} \int_{0}^{\infty} |\widehat{E}(\nu)|^2\, d\nu \ .$$

We can now identify the energy per unit area per unit band width:

$$\frac{dW}{dA\, d\nu} = \frac{c}{2\pi} |\widehat{E}(\nu)|^2 \ .$$

This is for the entire pulse and NOT per unit time. If, however, the pulse repeats on average time scale T, then we can write the mean flux per unit band width over this interval as

$$\langle S_\nu \rangle_T \equiv \left\langle \frac{dS}{d\nu} \right\rangle_T = \frac{dW}{dA\, d\nu\, dt} = \frac{c}{2\pi T} |\widehat{E}_\mathrm{T}(\nu)|^2 \ .$$

For very long signals which have more or less the same properties over their entire length, then

$$\langle S_\nu \rangle \equiv \left\langle \frac{dS}{d\nu} \right\rangle = \frac{c}{2\pi} \lim_{T\to\infty} \frac{|\widehat{E}_\mathrm{T}(\nu)|^2}{T}$$

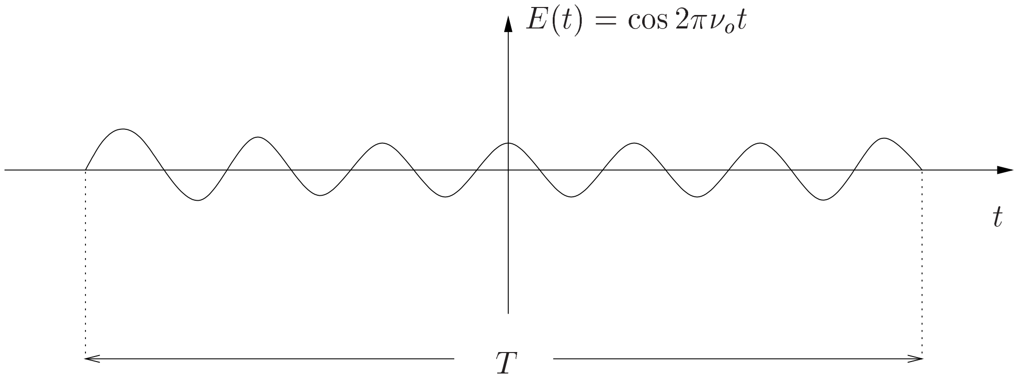
where $\widehat{E}_\mathrm{T}(\nu)$ is the transform of a *portion* of the wave train $E(t)$ over the interval T.

Frequency Resolution and the Uncertainty Principle

Note that the spectrum of the signal $E(t)$ cannot be defined at a given instant ($T \to 0$). Fourier transform theory shows that the width of the energy flux spectrum is $\Delta\nu \geq 1/T$ where T is the duration of the signal or pulse. Hence, if the signal is very brief in time, it's very broad in frequency. This is like an uncertainty principle:

$$\Delta\nu\Delta t \geq 1 \ .$$

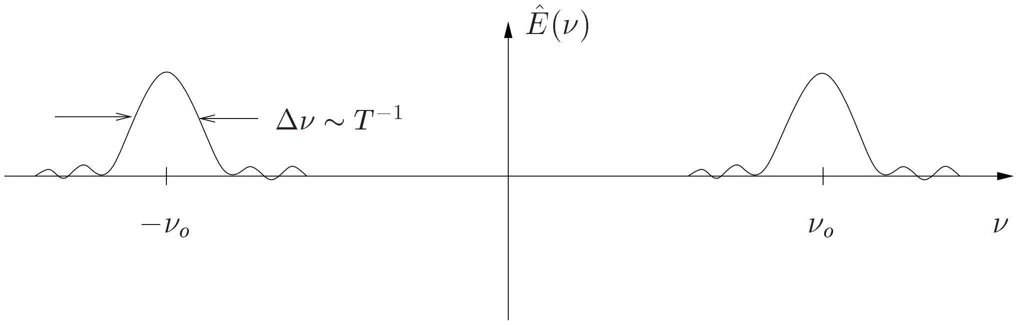
This is easy to prove for a monochromatic wave train of frequency ν_0 and duration T.



I leave it as an exercise to show that the Fourier transform of the foregoing signal is

$$\widehat{E}(\nu) = \frac{\sin \pi(\nu - \nu_0)T}{2\pi(\nu - \nu_0)} + \frac{\sin \pi(\nu + \nu_0)T}{2\pi(\nu + \nu_0)} \ .$$

In graphical form, this looks like



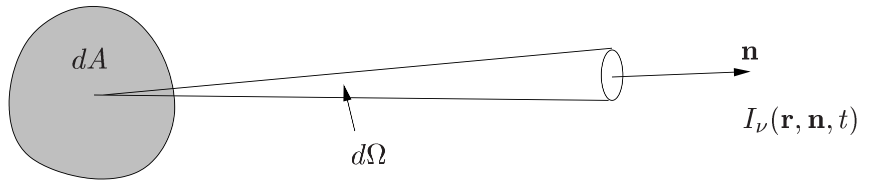
Clearly the spectrum of the monochromatic signal is broadened in frequency to $\Delta\nu \sim 1/T$. One cannot distinguish two frequencies present in the signal if they are separated by less than $1/T$ Hz.

1.4 Radiative Transfer

We start by introducing some basic quantities from classical radiometry theory which is based on a fluid description of light. These quantities are measures of the *macroscopic* properties of the radiation field.

(a) Specific Intensity (I_ν) or Brightness (B_ν)

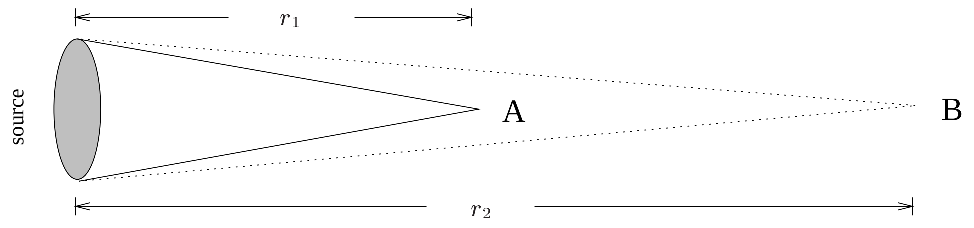
I_ν measures the *angular* distribution of energy flow in a radiation field. Consider a particular beam or ray direction $\boldsymbol{n}$ normal to an element of area dA:



The energy dE passing through dA within solid angle $d\Omega$ in time dt and in the frequency interval or bandwidth $d\nu$ is

$$dE_\nu = I_\nu(\boldsymbol{r}, \boldsymbol{n}, t)\, dA\, dt\, d\Omega\, d\nu \ .$$

This can be considered the definition of the specific intensity. Clearly, I_ν is the flux per unit solid angle per unit bandwidth; it has units of ergs $\mathrm{s}^{-1}\,\mathrm{cm}^{-2}\,\mathrm{ster}^{-1}\,\mathrm{Hz}^{-1}$. The "per unit solid angle" property makes I_ν invariant along a given direction, unlike the flux which falls off like r^{-2}. This assumes that there are no sinks or sources of energy along the beam.



At point B, the flux is smaller than at A by a factor $(r_1/r_2)^2$ but the solid angle subtended by the source is also smaller by exactly the same factor. Therefore, the energy per unit solid angle (I_ν) is exactly the same at both points. Obviously, if there were a source (or sink) of energy between A and B, that would no longer be true.

(b) Flux (or Flux Density): $\boldsymbol{F}_\nu$

This is a *vector* quantity; it describes the first moment of the radiation field or specific intensity:

$$\boldsymbol{F}_\nu(\boldsymbol{r}, t) = \oint I_\nu(\boldsymbol{r}, \boldsymbol{n}, t)\, \boldsymbol{n}\, d\Omega \ ,$$

where the integral is over all solid angle: $d\Omega \equiv \sin\theta\, d\theta\, d\varphi$ in spherical polar coordinates. In the same coordinates, the unit vector $\boldsymbol{n}$ becomes

$$\boldsymbol{n} = \mathbf{1}_x \sin\theta\, \cos\varphi + \mathbf{1}_y \sin\theta\, \sin\varphi + \mathbf{1}_z \cos\theta \ .$$

The flux is the vector sum of all the beams passing through a particular point. If I_ν is isotropic (i.e., I_ν independent of $\boldsymbol{n}$), then there is obviously no net flux: $\boldsymbol{F}_\nu = 0$.

(c) Energy Density: U_ν

This is the energy per unit volume per unit band width in the radiation field. Consider energy flowing in a particular direction. We saw earlier that the time averaged energy flux $\langle S\rangle = cU_{\text{rad}}$. If we now interpret $\langle S\rangle$ to be an infinitesimal contribution $I_\nu\, d\Omega$ to the flux contained in a small solid angle $d\Omega$, then the corresponding infinitesimal contribution to the energy density is

$$dU_\nu = \frac{1}{c} I_\nu\, d\Omega \ .$$

The *total* energy density is found by integrating over 4π steradians:

$$U_\nu(\boldsymbol{r}, t) = \frac{1}{c} \oint I_\nu(\boldsymbol{r}, \boldsymbol{n}, t)\, d\Omega \ .$$

(d) Mean Intensity: J_ν

This is the zeroth moment of the radiation field:

$$J_\nu(\boldsymbol{r}, t) = \frac{1}{4\pi} \oint I_\nu(\boldsymbol{r}, \boldsymbol{n}, t)\, d\Omega \ .$$

Obviously, $U_\nu = 4\pi J_\nu / c$.

The second moment of the radiation field

$$\oint I_\nu(\boldsymbol{r}, \boldsymbol{n}, t) \boldsymbol{n}\boldsymbol{n}\, d\Omega$$

is also a useful physical quantity. It's related to the radiation pressure tensor. Computing its *nine* elements for an isotropic I_ν is a useful exercise that you might like to try.

Equation of Radiative Transfer

Radiative transfer theory describes the change in I_ν along a particular path due to *absorption* and *emission*. Introduce the macroscopic coefficients:

(a) $j_\nu \equiv$ emission coefficient = energy emitted per unit time, per unit volume, per unit solid angle, per unit band width

If the emission is spontaneous only, then j_ν is generally isotropic. However, if it includes scattering, then j_ν can be nonisotropic, reflecting the local radiation field.

(b) $\alpha_\nu \equiv$ absorption coefficient = absorption cross-section per unit volume (it is measured in units of cm^{-1}).

In the absence of emission,

$$dI_\nu \stackrel{\alpha_\nu}{\equiv} -\alpha_\nu I_\nu ds \ .$$

This phenomenological relation assumes that the linear scale of the cross-section per particle is much smaller than the interparticle distance. Also, the absorbers act independently and are randomly distributed. (Note that instead of α_ν, we could use the *atomic* or *mass* absorption coefficients: $\alpha_\nu = N\sigma_\nu = \rho k_\nu$.)

With these definitions, it is straightforward to write down the change in I_ν along a path of length ds:

$$dI_\nu = -\alpha_\nu I_\nu \, ds + j_\nu \, ds$$

or

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \ .$$

Now introduce the *optical depth* $d\tau_\nu \equiv \alpha_\nu \, ds$ and the *source function* $S_\nu \equiv j_\nu/\alpha_\nu$. Then the transfer equation transforms to its more standard form

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu \ .$$

Note that the optical depth is defined only with respect to *two* points. In that regard, it's like a *potential*:

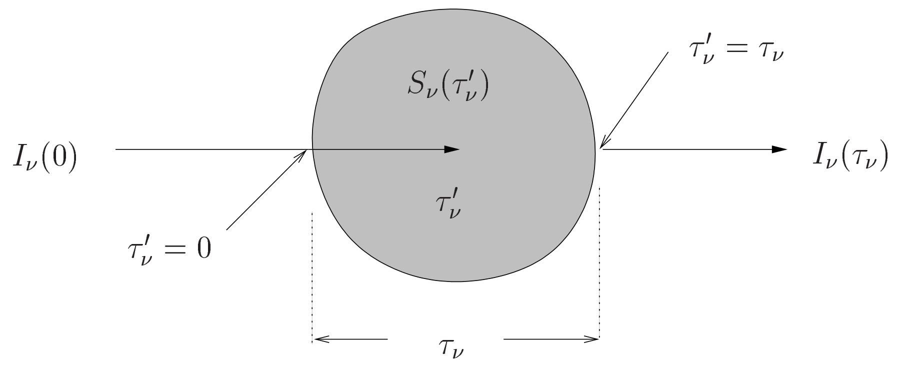
$$\tau_\nu(s) = \int_0^s \alpha_\nu(s') \, ds' \ .$$

The optical path between $s' = 0$ and $s' = s$ is *opaque* (optically thick) when $\tau_\nu > 1$. It's *transparent* (optically thin) when $\tau_\nu < 1$.

Now the differential transfer equation can be easily converted into an integral transfer equation by multiplying through by an integrating factor e^{τ_ν}:

$$\mathrm{e}^{\tau_\nu} \left(I_\nu + \frac{dI_\nu}{d\tau_\nu} \right) = \frac{d}{d\tau_\nu} \left(\mathrm{e}^{\tau_\nu} I_\nu \right) = S_\nu \mathrm{e}^{\tau_\nu} \ .$$

Now integrate between $\tau'_\nu = 0$ and $\tau'_\nu = \tau_\nu$:



$$\mathrm{e}^{\tau_\nu} I_\nu(\tau_\nu) - I_\nu(0) = \int_0^{\tau_\nu} S_\nu(\tau'_\nu)\,\mathrm{e}^{\tau'_\nu}\,d\tau'_\nu$$

or

$$I_\nu(\tau_\nu) = I_\nu(0)\,\mathrm{e}^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\tau'_\nu)\,\mathrm{e}^{-(\tau_\nu-\tau'_\nu)}\,d\tau'_\nu\ .$$

This is the formal solution to the equation of transfer. Its physical interpretation is fairly straightforward: the change in I_ν along the path τ_ν is the sum of all the contributions at each point on the path weighted by the attenuation suffered in going through the absorbing medium to the observation point. This "formal" solution is not of much use unless the source function S_ν is known. But if it is, then the specific intensity can be computed at any point and in any direction.

Consider the special case where $S_\nu(\tau'_\nu)$ is a constant (S_ν, say):

$$I_\nu(\tau_\nu) = I_\nu(0)\,\mathrm{e}^{-\tau_\nu} + S_\nu(1 - \mathrm{e}^{-\tau_\nu})\ .$$

The opaque and transparent limits are

(i) $\tau_\nu \to \infty$: $\quad I_\nu(\tau_\nu) \to S_\nu$;

(ii) $\tau_\nu \to 0$: $\quad I_\nu(\tau_\nu) \to I_\nu(0) + S_\nu\tau_\nu \to I_\nu(0)$.

Note that we have demonstrated here that I_ν is constant along a path with no absorption or emission; i.e., $dI_\nu/ds = 0$ if $\alpha_\nu = j_\nu = 0$. If there is absorption but no emission, then I_ν falls off like $\mathrm{e}^{-\tau_\nu}$. After one optical depth, I_ν is down by a factor e^{-1}. This is the probability that a photon will be absorbed or scattered in one optical depth.

Another important point is that the observer "sees" the source function only in the optically thick limit. In the optically thin case, he sees the *emission coefficient* instead:

$$I_\nu(\tau_\nu) \to S_\nu\tau_\nu \simeq j_\nu\,L \quad (\tau_\nu \to 0)$$

where L is the physical scale or dimension of the source.

1.5 Blackbody Radiation and Thermal Emission

Thermal emission is radiation emitted by matter in thermodynamic equilibrium or *local* thermodynamic equilibrium. The latter state exists when the temperature is effectively constant over a photon mean free path (or one optical depth). A stellar interior is a good example of a region in LTE. Thermal emission is also possible in situations where the gas is in thermodynamic equilibrium with itself but not necessarily with the local radiation field. A good example of this situation is the stellar photosphere of a main sequence star where collisions are frequent enough to maintain a Boltzmann distribution of atomic states; this ensures that $S_\nu = B_\nu$. Thus, the gas can radiate like a blackbody even though the region as a whole is not in thermodynamic equilibrium.

The Blackbody Spectrum (B_ν)

Divide the problem of calculating the spectrum into two parts:

(a) find the density of energy states,

(b) evaluate the mean energy per state.

The product of these two quantities at a particular frequency ν gives the energy density at that frequency.

(a) Density of States

There are different ways of making this calculation. In classical statistics (Debye theory), one identifies photon energy states with the number of standing wave nodes in an enclosure of arbitrary volume and shape:

$$\begin{array}{l}\#\text{ of standing wave nodes/unit volume} \\ \text{in frequency interval } d\nu\end{array} = \frac{4\pi\nu^2 d\nu}{c^3} \ .$$

Multiply this expression by a factor 2 to allow for the two independent planes of polarization. (See Rybicki & Lightman p20.) A more transparent (??) approach is to invoke a basic tenet of quantum mechanics: phase space is quantized into phase cells of volume h^3. The number of phase cells between ν and $\nu + d\nu$ per unit volume of real space is

$$dN_\nu = \frac{8\pi p^2\, dp}{h^3} = \frac{8\pi\nu^2\, d\nu}{c^3} \qquad (p \equiv h\nu/c)$$

where I have thrown in a factor 2 as before. This actually gives the *maximum* density of electrons in phase space because Fermi statistics limits them to *two* per phase cell. But photons are bosons and, in principle, there is no limit on how many can occupy each phase cell. But we can calculate the mean number and, hence, the mean energy of each state.

(b) Mean Energy per State

Let's assume that a particular state at frequency ν contains n photons. The energy in this state will be $E_n = nh\nu$. According to the Boltzmann law, the probability that a state will have this energy in thermodynamic equilibrium is proportional to $\mathrm{e}^{-E_n/kT}$. Therefore, the *average* energy per state should be

$$\bar{E} = \frac{\sum_0^\infty E_n \mathrm{e}^{-E_n/kT}}{\sum_0^\infty \mathrm{e}^{-E_n/kT}} = -\frac{\partial}{\partial(kT)} \ln\left(\sum_0^\infty \mathrm{e}^{-E_n/kT}\right) \ .$$

The sum here is a geometric series which can be easily evaluated if one recalls that $\sum_0^\infty ar^n = a/(1-r)$:

$$\sum_0^\infty \mathrm{e}^{-E_n/kT} = \sum_0^\infty \mathrm{e}^{-nh\nu/kT} = \frac{1}{(1-\mathrm{e}^{-h\nu/kT})} \ .$$

Hence,

$$\begin{aligned} \bar{E} &= -\frac{\partial}{\partial(kT)} \ln(1-\mathrm{e}^{-h\nu/kT})^{-1} \\ &= \frac{1}{(1-\mathrm{e}^{-h\nu/kT})} \ . \ \mathrm{e}^{-h\nu/kT} \ . \ h\nu \\ &= \frac{h\nu}{(\mathrm{e}^{h\nu/kT}-1)} \end{aligned}$$

which goes to kT as $h\nu \to 0$. It is now obvious that the average number of photons per energy state or the *occupation number* n_ν is

$$n_\nu = \left(\mathrm{e}^{h\nu/kT}-1\right)^{-1}$$

which goes asymptotically to $kT/h\nu$ for small ν. The number of photons between ν and $\nu + d\nu$ will be n_ν multiplied by the number of states in that frequency range; this is

$$n_\nu \, dN_\nu = n_\nu \, \frac{8\pi\nu^2 \, d\nu}{c^3}$$

or the Bose-Einstein number density for a "chemical potential" (η) of zero. Thus photons are distributed in frequency (energy) like the Boltzmann Law only for *high* ν where the photon density is relatively low.

We can now combine the mean energy per photon at frequency ν with the number of photons per unit volume at frequency ν to obtain

$$U_\nu \, d\nu = h\nu \, n_\nu \, dN_\nu = \frac{h\nu}{(\mathrm{e}^{h\nu/kT}-1)} \, \frac{8\pi\nu^2 \, d\nu}{c^3}$$

or

$$U_\nu = \frac{8\pi h\nu^3}{c^3} \left(\mathrm{e}^{h\nu/kT}-1\right)^{-1} \ .$$

To get the specific intensity, $I_\nu = B_\nu$ in this case, use

$$U_\nu = \frac{4\pi B_\nu}{c} \quad \text{or} \quad B_\nu = \frac{c}{4\pi} U_\nu \ .$$

One more step yields the blackbody spectrum:

$$B_\nu = \frac{2h\nu^3}{c^2} \left(\mathrm{e}^{h\nu/kT}-1\right)^{-1} = -B_\lambda \frac{d\lambda}{d\nu} \ .$$

Properties of the Planck Function

(a) $h\nu \ll kT$: *Rayleigh-Jeans Law*

Note that this is the regime of *large* occupation number ($n_\nu \gg 1$). Approximate e^x by $1+x$ (small x) to obtain

$$B_\nu \simeq 2kT\,\frac{\nu^2}{c^2} = \frac{2kT}{\lambda^2} \quad \text{(dominated by photon statistical weight)} .$$

This law applies to the frequency range explored by radio astronomy.

(b) $h\nu \gg kT$: *Wien's Law*

In the regime of *small* occupation number, on the other hand,

$$B_\nu = \frac{2h\nu^3}{c^2}\,e^{-h\nu/kT} \quad \text{(dominated by Boltzmann Law)} .$$

(c) *Temperature Dependence*

The foregoing laws show that at any given frequency, $B_\nu \to \infty$ as $T \to \infty$ and $B_\nu \to 0$ as $T \to 0$. Furthermore, it is easy to show that $\partial B_\nu/\partial T > 0$ which says that B_ν increases monotonically with temperature at a given frequency.

(d) *Wien Displacement Law*

The frequency $\nu_{\rm max}$ at which the peak of $B_\nu(T)$ occurs can be found by solving $\partial B_\nu/\partial \nu = 0$; one finds

$$h\nu_{\rm max} = 2.82\,kT \ .$$

(e) *Stefan-Boltzmann Law*

First, find the integrated Planck function:

$$B(T) = \int_0^\infty B_\nu(T)\,d\nu = \frac{2\pi^4 k^4}{15c^2h^3}T^4 = \frac{\sigma}{\pi}T^4$$

where $\sigma \equiv ac/4$ is the Stefan-Boltzmann constant and a is the radiation density constant. Since $U_\nu = 4\pi B_\nu/c$, we can easily obtain

$$U(T) = aT^4 \ .$$

Next, find the integrated flux per unit area from the surface of a blackbody:

$$\begin{aligned} F &= \oint_{2\pi} B(T)\,(\mathbf{1}_z \cdot \boldsymbol{n})\,d\Omega \\ &= B(T)\int_0^{2\pi} d\varphi \int_0^{\pi/2} \cos\theta\,\sin\theta\,d\theta \\ &= B(T)\,.\,2\pi\,.\,\frac{1}{2} \\ &= \pi B(T) \end{aligned}$$

The final step yields the Stefan-Boltzmann Law:

$$F = \sigma T^4$$

where

$$\begin{aligned} \sigma &= \frac{2\pi^5 k^4}{15 c^2 h^3} = 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{ deg}^{-4} \text{ s}^{-1} \ , \\ a &= \frac{4\sigma}{c} = 7.56 \times 10^{-15} \text{ erg cm}^{-3} \text{ deg}^{-4} \ . \end{aligned}$$

Characteristic Temperatures

(a) *Brightness Temperature*: T_b

Defined by assuming that the observed flux comes from a blackbody:

$$B_\nu(T_b) = I_\nu(\text{obs}) \ .$$

In the radio or Rayleigh-Jeans limit, $B_\nu = 2kT/\lambda^2$ which results in

$$T_b = \frac{\lambda^2}{2k} I_\nu(\text{obs}) = \frac{c^2}{2\nu^2 k} I_\nu(\text{obs}) \ .$$

(b) *Color Temperature*: T_c

Defined by a fit to the Planck curve (or portion thereof). In some cases, knowing the temperature of a source will allow the observer to estimate its absolute luminosity and, hence, its distance. This works reasonably well for main sequence stars for which effective temperatures can be inferred from spectral types with some degree of confidence. This method is sometimes referred to as *spectroscopic parallax.*

(c) *Effective Temperature*: T_{eff}

Defined by a fit to the Stefan-Boltzmann law:

$$F_{\text{obs}} = \sigma T_{\text{eff}}^4 \left(\frac{R_\star^2}{r^2} \right)$$

or

$$L = 4\pi R_\star^2 \, \sigma T_{\text{eff}}^4 = 4\pi r^2 \, F_{\text{obs}}$$

for stars with a radius $R_\star$ at a distance r.

Note that T_b and T_{eff} require a knowledge of the *absolute* source intensity (i.e., the distance) but T_c depends only on the *shape* of the observed spectrum and is, therefore, independent of distance.

1.6 Thermodynamic Equilibrium

See Chapters 6–8 (pp. 46-76) of Shu for a rather detailed discussion of thermodynamic equilibrium.

Summary

A closed system of fixed energy left to itself will eventually attain a state of thermodynamic equilibrium. Its properties are as follows:

(a) *entropy* reaches a maximum for the particular temperature which is spatially constant;

(b) *detailed balancing* obtains – every elementary process is statistically balanced by its exact inverse;

(c) *a Boltzmann distribution* of energy states is established, at least for sufficiently low densities where the Pauli Exclusion Principle is not relevant:

$$\frac{n_2}{n_1} = \frac{g_2}{g_1}\,\mathrm{e}^{-(E_2-E_1)/kT}$$

where g_1 and g_2 are statistical weights which are basically the number of ways of populating or occupying the given states. Note that blackbody radiation is a direct consequence of this basic law. The statistical weights in this case are the density of states within a certain range in frequency. The probability of n photons occupying a state of frequency ν is

$$P_n \propto \mathrm{e}^{-E_n/kT} = \mathrm{e}^{-nh\nu/kT} \ .$$

The Boltzmann distribution of particle velocities is another consequence of the Boltzmann law:

$$P(v)\,dv \propto 4\pi v^2\,\mathrm{e}^{-mv^2/2kT}dv$$

where $4\pi v^2\,dv$ is effectively the statistical weight of a free particle; it is the volume of velocity space (per unit volume of real space) available to a particle with velocity v. Saha's equation which governs the distribution of atomic ionization states can also be shown to follow directly from the Boltzmann law.

Thus, these fundamental distributions which describe thermodynamic equilibrium are all based on this *most probable* energy distribution law. One should keep in mind, however, that the Boltzmann distribution is not so fundamental that it is universal; it will break down at high densities where particles must follow Fermi-Dirac statistics and not Maxwell-Boltzmann statistics which is just an approximation that is valid at low densities.

(d) The radiation field is isotropic implying no net flux everywhere.

These are characteristics of "Strict Thermodynamic Equilibrium" or STE, a state which never exists in real astrophysical situations. What one finds, instead, is "Local Thermodynamic Equilibrium" or LTE which prevails in regions that are optically thick enough to trap radiation so that it can come (almost) into equilibrium with the matter. Stellar interiors are good examples of regions in LTE: there is a small temperature gradient, a small anisotropy in the radiation field, and a small flux. But the gas still radiates like a blackbody corresponding to a Boltzmann distribution of atomic states and a Maxwell-Boltzmann velocity distribution.

1.7 Kirchoff's Law for Thermal Emission

Recall that

$$I_\nu = I_{\nu o}\,\mathrm{e}^{-\tau_\nu} + S_\nu(1 - \mathrm{e}^{-\tau_\nu}) \ .$$

Inside a blackbody cavity,

$$I_\nu = I_{\nu o} = B_\nu \ .$$

Therefore,

$$B_\nu = B_\nu\,\mathrm{e}^{-\tau_\nu} + S_\nu(1 - \mathrm{e}^{-\tau_\nu})$$

or

$$B_\nu(1 - \mathrm{e}^{-\tau_\nu}) = S_\nu(1 - \mathrm{e}^{-\tau_\nu})$$

or

$$S_\nu = B_\nu \qquad \Longrightarrow \qquad j_\nu = \alpha_\nu B_\nu \ .$$

Here, we can draw the distinction between blackbody radiation, $I_\nu = B_\nu$, and thermal emission, $S_\nu = B_\nu$. The latter relation also follows directly from detailed balancing as we shall see later.

It is worth noting that $S_\nu = B_\nu$ can apply even if the gas is not in thermodynamic equilibrium. This was pointed out earlier. We need only require that the distribution among energy states be equivalent to that in equilibrium; namely, Boltzmann. This can happen, for example, at the photospheres of stars where the gas can be collisionally dominated. Thermal emission also applies in LTE as inferred on the preceding page. In a stellar interior, for example, the change in temperature over one mean free path of the photon, ΔT, satisfies $\Delta T/T \ll 1$ or $dT/d\tau \ll T$. This means that all the photons passing through a point in a stellar interior come from regions with the same temperature (to within ΔT – a measure of how accurately the temperature can be defined). In this situation, $S_\nu = B_\nu$ but $I_\nu \neq B_\nu$.

Case Study: Determining the Dust Mass in External Galaxies

The grains of dust which inhabit the ISM of galaxies have radii $a \leq 0.25\,\mu$m ($1\,\mu\mathrm{m} = 10^{-4}$ cm) and a grain temperature T_g from 20-2000 °K depending on their proximity to stellar radiation fields. Note that $\lambda_{\max}$ from Wien's Law is around $100\,\mu$m for $T \sim 30\,$°K.

The solid dust grains are in equilibrium and radiate like black bodies: $S_\nu = B_\nu(T_g)$. Therefore, neglecting background emission and assuming that the dust is optically thin in the infrared, we can write

$$I_\nu = \int B_\nu(T_g)\,d\tau_\nu = \int \rho\kappa_\nu B_\nu\,ds \qquad (\tau_\nu \ll 1) \ .$$

[Note that I am using here the *thermal* absorption coefficient per unit mass κ_ν (rather than the *total* absorption coefficient k_ν which includes scattering) because we are estimating the magnitude of the *thermal* emission).]

Now assume that the galaxy subtends a relatively small solid angle $d\Omega = dA/D^2$. Then

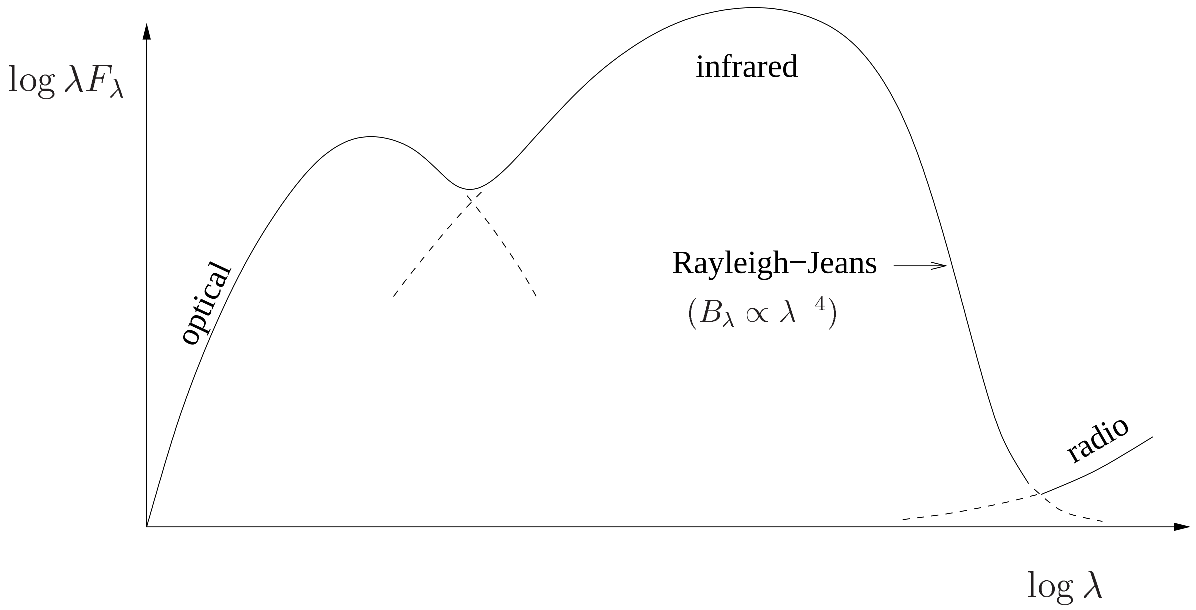
$$F_\nu = I_\nu\,\frac{dA}{D^2} = \frac{\kappa_\nu B_\nu}{D^2}\int \rho\,dA\,ds = \frac{M}{D^2}\,\kappa_\nu B_\nu(T_g) \ ,$$

where M = total mass in the form of dust. Now

$$\kappa_\nu(\mathrm{dust}) = n_g Q_\nu \sigma$$

where n_g is number density of grains (per unit mass), Q_ν is the absorption efficiency (proportional to νa for small ν), and $\sigma \simeq \pi a^2$. Note that at these infrared wavelengths ($\lambda \gg a$) the grain opacity depends only on grain *volume* and not on *size*. Therefore, in this case, $\kappa_\nu \propto \nu/\rho_g$ where the grain mass density $\rho_g \sim$ few $\mathrm{gm\,cm^{-3}}$ depending on the composition. Thus, extinction is very low in the infrared unlike in the visible frequency range. This ability to penetrate the "murk" is the big advantage of IR observations.

Thus, we have $F_\nu \propto \nu B_\nu$ or $\lambda F_\lambda \propto B_\lambda$. In a typical source, one might see something like this:



IR Excess in NGC6240

See "*Dust in External Galaxies*" by Thronson & Skull (Eds.) and "*The Interstellar Medium in Galaxies*", p 483, by Draimè.

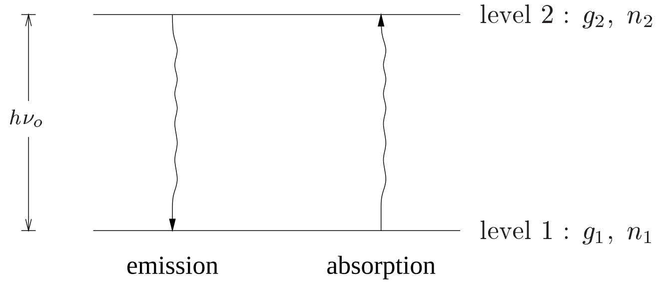
This exercise of estimating the dust mass in a source is complicated by the uncertain composition of the grains (which affects κ_ν), the wavelength dependence of κ_ν at long wavelength, and the number of dust components (at different T) in the galaxy in question. However, usually most of the IR excess comes from the coolest component (20-30 °K). One can try to fit multiple components to observations (such as those shown for NGC6240) or just go for the mass of the coolest component. One finds typically that $M_{\text{dust}} \sim 10^{-4} - 10^{-3}\, M_{\text{galaxy}}$. Despite its low mass, however, dust is an important component of galaxies for a number of reasons including

- many of the heavy elements of the interstellar gas are locked up in dust grains;
- it is an important source of cooling for the gas at low temperatures along with molecular hydrogen;
- it presents a large cross-section for absorption (extinction) which means that dust interacts strongly with radiation (at least at shorter wavelengths), explaining its importance in the driving of winds and the limiting of accretion onto young stars.

1.8 Einstein Coefficients

The Einstein coefficients characterize or govern the *microscopic* processes of emission and absorption which are implicit in Kirchoff's Law. To make things simple, let's consider atomic transitions between *bound* states. But keep in mind that much of the formalism can be carried over to continuum states and many of the conclusions apply equally well in that limit.

Consider two states separated in energy by $h\nu_0$:



In this figure, g_1 and g_2 are *statistical weights* which are measures of the number of electrons that can fit into these states per atom or molecule. For example, the statistical weight of the n^{th} level of H is $2n^2$ which is obtained by adding up all the angular momentum and spin substates for a given n.

There are *three* fundamental processes corresponding to the three types of transitions between these two states:

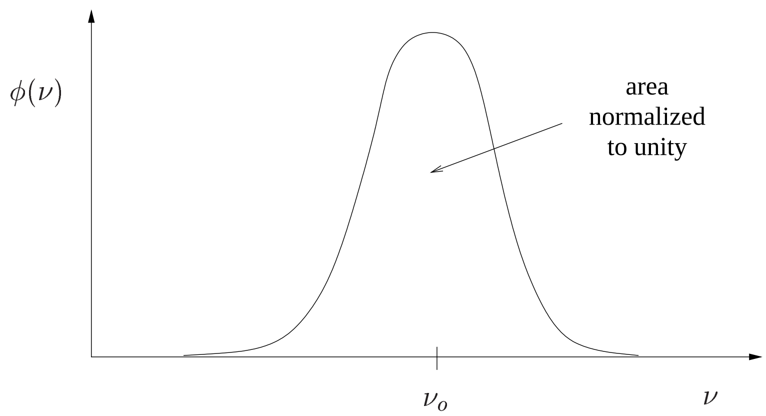
(a) <u>spontaneous emission (characterized by A_{21})</u>:

$$A_{21} \equiv \text{transition probability per unit time for spontaneous emission.}$$

Thus, the downward transition rate per unit volume is $n_2 A_{21}$.

(b) <u>absorption (characterized by B_{12})</u>:

The upward transition rate per unit volume induced by the presence of the mean intensity $\bar{J}$ is $n_1 B_{12} \bar{J}$. The bar on J signifies an average across the spectral line profile $\phi(\nu)$ which describes the effectiveness of the absorption as a function of ν:



The area under the curve here has been normalized to unity; i.e., $\int \phi(\nu)\,d\nu = 1$. Thus, the mean intensity by this definition is

$$\bar{J} \equiv \int_0^\infty J_\nu\,\phi(\nu)\,d\nu\ .$$

(c) stimulated emission (characterized by B_{21}):

Einstein found that there is also a need to invoke a process of stimulated emission whereby there are downward transitions induced by the stimulation of the atom by external photons:

$$n_2 B_{21} \bar{J} \equiv \text{downward transition rate per unit volume.}$$

Now the relationships between these three Einstein coefficients can be derived by assuming thermodynamic equilibrium. But the relations obtained in this way are valid for *all* physical conditions because the coefficients are properties of the atom and not of its environment.

Thus, in equilibrium,

$$n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}$$

or

$$\bar{J} = \frac{A_{21}/B_{21}}{n_1 B_{12}/n_2 B_{21} - 1}\ .$$

But, in thermodynamic equilibrium, $\bar{J}$ must be equal to $B_\nu(\nu_0)$ and

$$\frac{n_1}{n_2} = \frac{g_1}{g_2}\,e^{h\nu_0/kT} \qquad \text{(Boltzmann Law)}\ .$$

Therefore,

$$B_\nu = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21})\,e^{h\nu_0/kT} - 1} = \frac{2h\nu^3/c}{(e^{h\nu_0/kT} - 1)}\ .$$

Identifying terms, we see that

$$g_1 B_{12} = g_2 B_{21} \quad \text{and} \quad A_{21} = \frac{2h{\nu_0}^3}{c^2}\,B_{21}\ .$$

These relationships (remember) are completely general and not dependent on thermodynamic equilibrium.

The Macroscopic Line Absorption and Emission Coefficients in Terms of the Atomic Einstein Coefficients

We drop the assumption of thermodynamic equilibrium but continue to assume an isotropic radiation field:

(a) Emission

From the definition of j_ν, it is clear that

$$\int_0^\infty j_\nu\,d\nu = n_2 A_{21}\,\frac{h\nu_0}{4\pi} \qquad \text{(spontaneous only)}\ .$$

This is satisfied by setting

$$j_\nu = \frac{h\nu_0}{4\pi}\,n_2 A_{21}\,\phi(\nu)$$

since $\int \phi(\nu)\,d\nu = 1$.

(b) Absorption

In the case of absorption, we recall that

$$dI_\nu = -\alpha_\nu I_\nu \, ds$$

or the energy absorbed per unit time per unit volume is

$$\begin{aligned} dE &= (n_1 B_{12} - n_2 B_{21}) \, \bar{J} \, h\nu_0 \\ &= \int_0^\infty \alpha_\nu \, d\nu \oint I_\nu \, d\Omega \\ &= 4\pi \int_0^\infty \alpha_\nu J_\nu \, d\nu \\ &= 4\pi \alpha_0 \bar{J} \ , \end{aligned}$$

where α_ν has been replaced by $\alpha_0 \phi(\nu)$. Therefore,

$$\alpha_0 = \frac{h\nu_0}{4\pi} (n_1 B_{12} - n_2 B_{21}) \ .$$

or

$$\alpha_\nu = \frac{h\nu_0}{4\pi} (n_1 B_{12} - n_2 B_{21}) \, \phi(\nu) \ .$$

Notice that this expression gives the *net* absorption coefficient allowing for stimulated emission. Remember that α_ν has units of *area per unit volume* or cm^{-1}.

An alternative form for α_ν is obtained as follows:

$$\alpha_\nu = \frac{h\nu_0}{4\pi} n_1 B_{12} \left[1 - \frac{n_2 B_{21}}{n_1 B_{12}}\right] \phi(\nu) \ .$$

Now

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \, \mathrm{e}^{-h\nu_0/kT_{\rm ex}} \qquad (T_{\rm ex} = T \text{ in thermodynamic equilibrium})$$

and

$$\frac{B_{21}}{B_{12}} = \frac{g_1}{g_2} \ .$$

Hence,

$$\alpha_\nu = \alpha_\nu^* \left(1 - \mathrm{e}^{-h\nu_0/kT_{\rm ex}}\right)$$

where α_ν^* is the absorption coefficient *uncorrected* for stimulated emission. Here, $T_{\rm ex}$ is, of course, the *excitation* temperature.

An interesting question is why are j_ν and α_ν defined in this way; namely, why does j_ν include only spontaneous emission. The above equations give one reason: convenience. Both absorption and stimulated emission have $\bar{J}$ in common whereas j_ν and α_ν don't depend on the mean intensity. If we defined j_ν differently, we would have

$$j_\nu = \frac{h\nu_0}{4\pi} n_2 \, (A_{21} + B_{21} \bar{J}) \, \phi(\nu)$$

which doesn't look so good! But the apparent neatness of the first option is misleading because $n_2 = n_2(\bar{J})$, in general.

Kirchoff's Law Revisited

Let's re-derive the source function S_ν:

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{n_2 A_{21}}{n_1 B_{12}(1 - e^{-h\nu/kT_{\rm ex}})}$$

where the common factor $\phi(\nu)$ cancels out. But

$$\frac{A_{21}}{B_{12}} = \frac{2h\nu^3}{c^2}\frac{g_1}{g_2} \quad \text{and} \quad \frac{n_2}{n_1} = \frac{g_2}{g_1}\,e^{-h\nu/kT_{\rm ex}} \ .$$

Therefore,

$$S_\nu = \frac{2h\nu^3}{c^2} \times \frac{g_1}{g_2} \times \frac{g_2}{g_1} \times \frac{e^{-h\nu/kT_{\rm ex}}}{(1 - e^{h\nu/kT_{\rm ex}})} = \frac{2h\nu^3}{c^2}\frac{1}{(e^{h\nu/kT_{\rm ex}} - 1)} = B_\nu(T_{\rm ex}) \ .$$

In other words, the source function is the Planck function evaluated at $T = T_{\rm ex}$. When thermodynamic equilibrium prevails, $T_{\rm ex} = T$ and $S_\nu = B_\nu(T)$ which recovers Kirchoff's Law. But if the system is out of equilibrium, then the excitation temperature will be different for different states (or frequencies).

Question: What if we redefined j_ν and α_ν by moving the stimulated emission from α_ν to j_ν? Would S_ν still equal B_ν in thermodynamic equilibrium?

Note too that Kirchoff's Law ($S_\nu = B_\nu$) requires only that a Boltzmann distribution of atomic states holds which implies a Maxwell-Boltzmann velocity distribution. That is, $T_{\rm ex} = T$ for *all* states (bound and free) and $T_{\rm kin} = T$. This can be true under a wide variety of circumstances even when strict thermodynamic equilibrium does not obtain, a point made earlier. It applies, in particular, to LTE where matter is in thermal equilibrium with the local gas but not necessarily in exact equilibrium with the local radiation field.

Particular Examples

- thermal emission from dust
- free-free emission from an interstellar gas cloud
- 21 cm emission from atomic hydrogen
- some molecules (e.g., CO at high densities)
- stellar photospheres (except for high mass stars & giants)
- stellar interiors (the best example)

The ff-emission and stellar photosphere cases hold due to the Maxwell-Boltzmann velocity distribution which often prevails in astrophysical situations and the fact that the gas is collisionally dominated. That is, the energy states (free and bound) are populated by collisions rather than by radiative transitions. In many environments, the particles have time to achieve an equilibrium velocity distribution because they are gravitationally bound. Photons, on the other hand, in the same situation are *not* bound and, therefore, it is often impossible for them to come into equilibrium with the local matter unless the gas is *optically thick*.

Scattering vs Absorption (Line and Continuum)

Scattering may be viewed as a *redirection* of photons by the medium, and hence the effective emission coefficient depends upon the incident intensity I_ν and not upon the internal state ($T_{\rm ex}$) of the gas.

In thermodynamic equilibrium, scattered radiation has the same spectrum as spontaneous emission (namely, B_ν) and it is not necessary to distinguish between them. But in non-TE conditions, the radiation field is, in general non-Planckian and nonisotropic. Therefore, it is necessary to treat absorption and scattering as separate processes and have a scattering coefficient, σ_ν as well as an absorption coefficient α_ν.

If we assume for the moment that the scattering is *conservative, coherent*, and *isotropic*, then the energy lost per unit volume by scattering into solid angle $d\Omega$ is

$$dE_\nu = \sigma_\nu I_\nu \frac{d\Omega}{4\pi} \ .$$

If we integrate this over all solid angles, then we can equate it to the scattering emission coefficient:

$$j_\nu^{(\rm scat)} = \frac{\sigma_\nu}{4\pi} \oint I_\nu \, d\Omega = \sigma_\nu J_\nu$$

to be compared with $j_\nu^{(\rm therm)} = \alpha_\nu B_\nu$. Thus, the scattering source function is

$$S_\nu^{(\rm scat)} = \frac{j_\nu^{(\rm scat)}}{\sigma_\nu} = J_\nu \neq B_\nu \quad \text{(in general)} \ .$$

We can easily incorporate scattering into the radiative transfer equation:

$$\frac{dI_\nu}{ds} = -(\alpha_\nu + \sigma_\nu)\, I_\nu + \alpha_\nu B_\nu + \sigma_\nu J_\nu$$

or

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

where

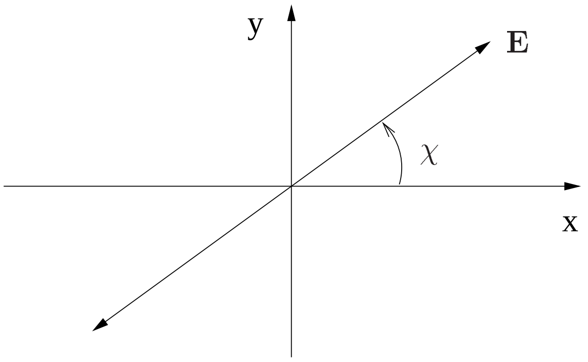
$$S_\nu = \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu} \ .$$

Note that this expression for the source function tends to B_ν as $J_\nu \to B_\nu$ in thermodynamic equilibrium. It is clear that this is a *mean* source function weighted by opacity. We are assuming here *thermal* emission and *coherent, isotropic*, and *conservative* scattering. Of course, under extreme non-LTE conditions, none of these assumptions need hold. For example, in the ISM most atoms and ions are in the ground state and, therefore, don't emit thermally. The main source of continuum scattering is *electron scattering* which is inherently nonisotropic and can be noncoherent as well at high frequencies (see the later discussion of Compton scattering). The other main scattering mechanism is *line scattering* from atoms and ions. This too can be noncoherent, nonisotropic, and even nonconservative.

The bottom line here is that scattering and thermal absorption are different processes and have to be treated separately unless the gas is in thermodynamic equilibrium. For example, the lines formed high in a stellar atmosphere (strong lines) are due mainly to scattering rather than to thermal absorption. Consequently, they can be completely black because $S_\nu^{(\rm scat)} = J_\nu$ which can be zero at a line frequency unlike B_ν.

1.9 Polarization and the Stokes Parameters

Electromagnetic waves may be *polarized*; that is, the $\boldsymbol{E}$-vector (and the $\boldsymbol{B}$-vector) may describe some coherent pattern in the plane normal to the direction of propagation. This means that their phases are correlated. The simplest example is *plane* or *linear* polarization. The vibration is at an angle χ to the x-axis. Propagation is out of the page. The amplitude is E_0. This behavior is characterized by a complete correlation between the x and y components.

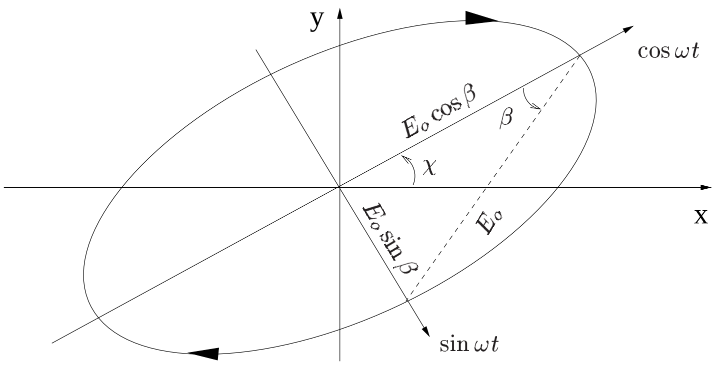


Similarly, one can have a *circularly* polarized wave in which case the x and y components are $\pi/2$ out of phase instead of in phase.

On the other hand, of course, radiation can be completely unpolarized and this is more generally the case. Most astrophysical sources emit photons which are uncorrelated with one another – their phases and planes of polarization are totally random. However, there are a few interesting situations (often involving magnetic fields) where the radiation can be at least partly polarized. Synchrotron radiation is, perhaps, the most common example. Also, electron scattering in nonspherical sources (e.g., rotating stars) can result in a small (nonzero) degree of polarization. Still other examples are aligned dust grains (which can polarize radiation), Zeeman splitting in magnetic fields, polarization of the CMB by gravity waves and, at the epoch of recombination, acoustic waves.

Monochromatic Waves

Waves of a given fixed frequency ω are the easiest to deal with. Since they are polarized, then the tip of the $\boldsymbol{E}$-vector will, in general, execute an *ellipse* inclined to the x-axis by an angle χ. In this figure, the beam is directed *out* of the page. Note that the state of polarization is characterized by E_0, β, and χ.



With the help of this figure, we can write

$$E_x = E_0(\cos\beta\,\cos\chi\,\cos\omega t + \sin\beta\,\sin\chi\,\sin\omega t)\ ,$$

and

$$E_y = E_0(\cos\beta\,\sin\chi\,\cos\omega t - \sin\beta\,\cos\chi\,\sin\omega t)\ .$$

Note that

$$-\frac{\pi}{2} \le \beta \le \frac{\pi}{2} \qquad \begin{cases} \beta = 0 \text{ or } \frac{\pi}{2} & \text{(linear polarization)} \\ 0 < \beta < \frac{\pi}{2} & \text{(left-handed polarization)} \\ -\frac{\pi}{2} < \beta < 0 & \text{(right-handed polarization)} \end{cases}$$

It is clear that we can also express E_x and E_y as

$$E_x = E_1 \cos(\omega t - \phi_1)\ , \qquad E_y = E_2 \cos(\omega t - \phi_2)$$

if we make the following identifications:

$$\begin{aligned} E_1 \cos\phi_1 &= E_0 \cos\beta\ \cos\chi\ , \\ E_1 \sin\phi_1 &= E_0 \sin\beta\ \sin\chi\ , \\ E_2 \cos\phi_2 &= E_0 \cos\beta\ \sin\chi\ , \\ E_2 \sin\phi_2 &= -E_0 \sin\beta\ \cos\chi\ . \end{aligned}$$

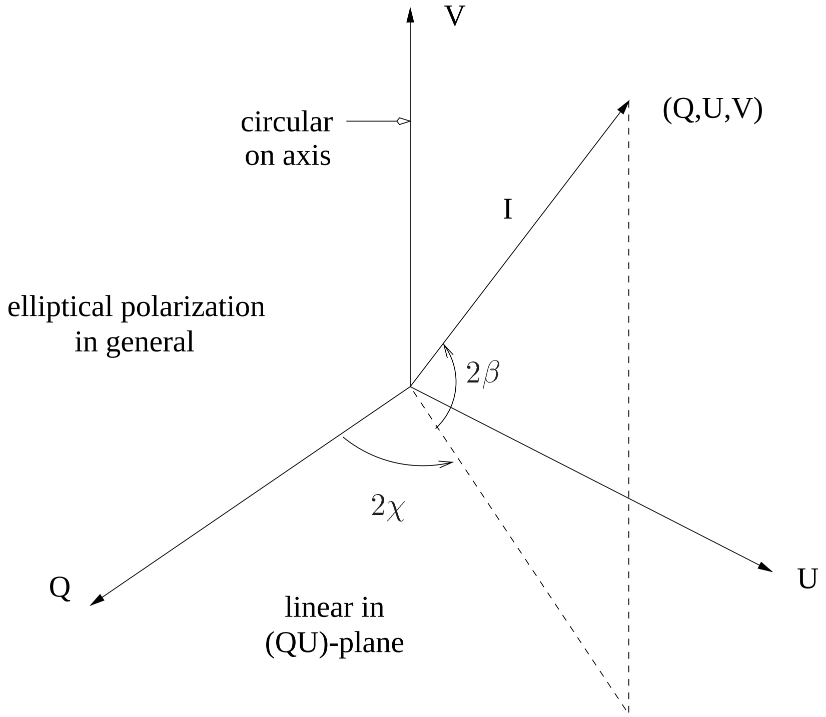
We are now in a position to define the four Stokes parameters:

$$\begin{array}{llll} I & \equiv & E_1{}^2 + E_2{}^2 = E_0{}^2 & \text{(total intensity)} \\ Q & \equiv & E_1{}^2 - E_2{}^2 = E_0{}^2\ \cos 2\beta\ \cos 2\chi & \text{(linear polarization)} \\ U & \equiv & 2E_1E_2\cos(\phi_1 - \phi_2) = E_0{}^2\ \cos 2\beta\ \sin 2\chi & \text{(linear polarization)} \\ V & \equiv & 2E_1E_2\sin(\phi_1 - \phi_2) = E_0{}^2\ \sin 2\beta & \text{(circular polarization)} \end{array}$$

from which we can extract

$$\begin{aligned} E_0 &= \sqrt{I}\ , \\ \tan 2\chi &= U/Q\ , \\ \sin 2\beta &= V/I\ , \\ I^2 &\equiv Q^2 + U^2 + V^2\ . \end{aligned}$$

Note that $(Q,\, U,\, V)$ can be interpreted as a *vector* of length I:



If the wave is linearly polarized ($\beta = 0,\ \pi/2$), then $V = 0$ and if the wave is circularly polarized ($\beta = \pm\pi/4$), then $Q = U = 0$.
For monochromatic waves, it is obvious that there are only *three* independent Stokes parameters.

Quasi-monochromatic Waves

In practice, we never see a single monochromatic component but rather a superposition of many components, each with its own polarization and phase. A special case occurs when the amplitudes and phases possess a *slow* time variation; that is, we have

$$E_x = E_1(t)\cos(\omega t - \phi_1(t)) \,, \quad \text{and} \quad E_y = E_2(t)\cos(\omega t - \phi_2(t)) \ .$$

We assume that over short times (of order $1/\Delta\omega$) the wave looks completely polarized with a definite $(Q,\, U,\, V)$. But over much longer times ($\Delta t \gg 1/\Delta\omega$), characterizing the timescale for changes in $E_1,\ E_2,\ \phi_1$, and ϕ_2, this state of polarization can change. Such a wave is no longer monochromatic; its bandwidth $\Delta\omega \sim 1/\Delta t \ll \omega$. For this reason, the wave is called "quasi-monochromatic" with a *coherence* time of Δt.

We may think of the wave having fluctuating Stokes parameters which might average to zero in a particular case. In any case, instead of a point in $(Q,\, U,\, V)$-space, the waves would occupy a finite sized region. The time-averaged Stokes parameters are given by

$$\begin{aligned} \bar{I} &= \langle {E_1}^2 + {E_2}^2 \rangle \ , \\ \bar{Q} &= \langle {E_1}^2 - {E_2}^2 \rangle \ , \\ \bar{U} &= 2\langle E_1 E_2 \cos(\phi_1 - \phi_2) \rangle \ , \\ \bar{V} &= 2\langle E_1 E_2 \sin(\phi_1 - \phi_2) \rangle \ . \end{aligned}$$

In this case, $E_1(t)$ and $E_2(t)$ might be only partly correlated or not correlated at all. Hence, $\bar{U}$ and $\bar{V}$ could become small or even *zero*. Thus, in general, $\bar{I}^2 \geq \bar{Q}^2 + \bar{U}^2 + \bar{V}^2$ and there are now *four* independent parameters. The equality sign applies to the case of complete polarization but, in general, we would have only *partial* polarization. The *degree* of polarization would be

$$\Pi \equiv (\bar{Q}^2 + \bar{U}^2 + \bar{V}^2)^{1/2} \,/\, \bar{I} \leq 1 \ .$$

In practice, most of the radiation we observe will be a superposition of independent or uncorrelated waves with different polarizations and phases. That is, a signal will have unpolarized and polarized components, the latter in turn having circular and linear polarized sub-components. An important property of the Stokes parameters is that they are *additive* for *uncorrelated* waves. [To see this, set $\boldsymbol{E} = \sum \boldsymbol{E}_n$ and remember that $\langle E_{n_1} E_{n_2} \rangle = 0$ for uncorrelated waves.] In fact, we can always decompose light from any astrophysical source into two parts: a completely unpolarized part ($\bar{Q}_u = \bar{U}_u = \bar{V}_u = 0$) and a part which is 100% polarized light ($\bar{Q}_p = \bar{Q}_{\rm obs},\ \bar{U}_p = \bar{U}_{\rm obs},\ \bar{V}_p = \bar{V}_{\rm obs}$). Then,

$$\bar{I}_p = (\bar{Q}^2 + \bar{U}^2 + \bar{V}^2)^{1/2} \ , \quad \bar{I}_u = \bar{I} - \bar{I}_p \ , \quad \text{and} \quad \Pi = \bar{I}_p \,/\, \bar{I} \ .$$

Special Case: $\bar{V}_p = 0$ (partial linear polarization)

Align the x-axis along the direction of polarization (so that $E_1 > E_2$). Then, for the polarized part, $\chi = 0$ and $\bar{U}_p = 0$. Thus, in this case,

$$\Pi = \frac{\bar{Q}_p}{\bar{I}} = \frac{\langle {E_1}^2 - {E_2}^2 \rangle}{\langle {E_1}^2 + {E_2}^2 \rangle} = \frac{I_{\rm max} - I_{\rm min}}{I_{\rm max} + I_{\rm min}} \ .$$

If circular polarization is present in a signal, then one can measure it by introducing a phase lag in one component; this converts the circular polarization into linear polarization which can be measured as indicated above. In the optical, one uses a quarter-wave plate to shift the phase of one component. In the radio, the phase lag is imposed electronically.