Star and Planet Formation Mini-Course
Assignment 2, Due Wednesday, April 13

1. Estimate the sensitivity of ground-based sub-mm telescopes to (optically thin) debris disks around main-sequence stars, in terms of disk mass, as a function of stellar type and distance. Consider specifically the sensitivity of the telescope APEX (currently in commission phase) and the future interferometer array ALMA (consisting of 64 APEX-like telescopes at up to 10 km baselines).

   For simplicity, you may estimate the sub-mm flux from a given disk by assuming that all dust is located at 100 AU with the number of grains $\mathrm{d}n(r) \propto r^{-\alpha}\mathrm{d}r$ where r is the size of the grain and $\alpha = 3.5$. Let the power-law be limited by the blow-out size of grains at the small end, and 100 km-size bodies at the large end. The albedo of the grains is 0.5, and the mass-luminosity relation for the central star is that of the main sequence.

2. Disk shepherding, gap opening and migration: The objective is to study the disk-planet interaction in a pressureless (or cold disk) approximation as if the gas parcels followed the trajectories of test particles. Consider a frame corotating with a planet of mass μM on a circular orbit. Here, μ is the mass ratio and M is the stellar mass.

   In that frame, we can use local approximation in which the nondimensional, radial difference of orbital radii of the planet (a_p) and a test particle or gas parcel (r) is small, $x = (r - a_p)/a_p \ll 1$. Locally, the disk shearing Keplerian motion introduces a flow with unperturbed velocity $v = -(1/2)\Omega a_p x$.

   (a) Consider now the approach of the test particle to the planet. Initial eccentricity of the incoming orbit is zero, right after encounter it jumps to e. Let us call the initial planet-particle orbit separation $d_0 = x(t = 0)$, and post-encounter d_1.

   Using the Jacobi integral of motion in the rotating frame of the planet, show that there are at least two possible outcomes. Either $e > 0$ and the planet pushes the smaller body or gas elements away from itself (shepherding of disk orbits), or alternatively the sign of x may flip, abs. value remain constant, and $e = 0$ even after the encounter (horseshoe orbits, U-turn orbits within the corotational zone extending out to 2.5 times the Roche lobe of the planet both in and out.)

   [Jacobi integral is derived in Binney and Tremaine 1990 textbook on Galactic Dynamics. It can be expressed as

   $$C = 1/r + 2\sqrt{r(1 - e^2)},$$

   the first being the negative of the specific energy of the test particle, the rest twice its specific angular momentum in the inertial frame (read BT if you're surprised! $C = const.$ along any given trajectory. Hint: expand it up to second order (keep terms $\sim x^2$) in the small parameter $x = (r - a_p)/a_p$, and write a relationship between d_1, d_0, and e).]

(b) Next, estimate how much eccentricity the test particle can gain in one encounter with the planet, using impulse approximation. In that approximation, following the spirit of 1st order perturbation theories, we follow the *unperturbed* straight-line trajectory, accumulating (time-integrating) the acceleration exerted on the body by the planet in the radial direction (x-axis component of Newtonian accel. by the planet.) Show that the time-integrated acceleration, i.e. the perpendicular velocity v_r is $v_r = 2GM\mu/(xv)$, where v is the Keplerian shear velocity (as above), and x is the initial orbital spacing

$v/(\Omega a_p)$ is then a good estimate of the orbital eccentricity gained in an encounter, since on the epicycle the maximum radial velocity does equal e times the local Keplerian speed. Now you should be able to write down the approximate expression for the jump in the orbital position, Δa of a body initially separated radially by xa_p from the planet. It should be a power-law in x, whose continuation to too small distances results in singularity. The singularity is removed by the so-called torque cutoff, which we will model by restricting x to values above a certain minimum value $|x| = x_s$ (separatrix distance).

(c) Now you are well equipped to tackle two important issues:

i. above what mass does a planet open a gap in a viscous disk?

ii. what is the type-I migration speed of the planet before it opens a gap?

For both tasks you need to obtain the estimate of the one-sided torque on the planet from the disk, i.e. angular momentum transfer rate from all the mass elements passing the star either inside or outside its corotational zone ($x > x_s$, say). Compute the angular momentum jump in one passage from the orbital separation jump, and take into account varying fluxes of incoming mass elements in a shearing medium. Assume that you know the disk surface density Σ and that it is constant on each side of the disk. Whenever possible, combine Σ into a dimensionless parameter $\mu_d = \pi a_p^2 \Sigma / M$.

However, for task (i) assume $x_s = 2.5 r_L/a_p$ (2.5 Roche lobe radii). Equate the gravitational gap-opening torque you derived with the viscous disk torque $3\pi\nu\Sigma\Omega a_p^2$ trying to close the and smooth the gap.

For task (ii) please assume $x_s = H$, the vertical disk scale height, since small bodies do not produce waves and do not couple well to the disk at smaller radial separations.

Finally, Goldreich and Tremaine (1980) argued that the differential torque from the imbalance of outer and inner disk torques is on the order of the one-sided torque times $H/a_p \approx 0.1$. This was later confirmed by others, to within a factor of order unity, and the inward direction was established.

By recalling the expression for specific angular momentum of a body, $L = \sqrt{GMa}$, you should now be able to write an expression for the rate of migration type I, the dangerous migration which, if unchecked by nature or overestimated by you, might remove and plunge to a fiery death the cores of growing planets in the solar nebulae all over this and other galaxies. So be a little careful.

What is the dependence of migration speed and timescale on mass ratio μ, and the disk thickness parameter, and what value do you obtain for 1 Earth mass (μ =3e-6, and 300 Earth masses (or one Jupiter, μ =0.001) in a standard solar nebula with $q_d = 0.002$ (at Jupiter's location)?

At what mass does a planet open a gap in a nebula with viscosity coefficient $\nu/(\Omega r^2) \approx 1e - 5$, corresponding to Shakhura-Sunyayev $\alpha \approx 1e - 2$? Can you sketch on a log-log diagram the migration rate and timescale for all planet masses, both pre-gap (type I) and gap-opening (type II) on one plot, knowing that the viscous speed of transport within an accretion disk (type II migr. speed) is $3\nu/(2r)$? (Please express speed in terms of Keplerian speed, not physical units. Assume we are interested in radial distances of order $r = 5$ AU to express timescale in years).

(d) Are there any ways for protoplanets to survive?