Midterm, 29 October 2012

Duration: 50 minutes

Examination aids: Anything except communication devices.

Notes: All (sub-)questions have equal weight. Answers can be brief and in point-form, but be sure that derivations can be followed.

1. We derive mass-radius relations for stars that are powered by nuclear fusion and thus have (almost) the same central temperature.
   (a) Use the Virial Theorem to show how radius scales with mass for balls of ideal gas that vary in mass but have the same central temperature.
   (b) For hydrogen fusion, a central temperature of $\sim 10^7\,$K is needed, while helium fusion requires $\sim 10^8\,$K. Physically, why does helium fusion require a much higher temperature?
   (c) Now compare a normal main-sequence star like the Sun with a $M = 1 M_\odot$ star composed of pure helium that is fusing helium in its core. Will this helium star be larger or smaller than the normal star? Approximately by how much? (*Somewhat more massive "helium main-sequence" stars exist – they are called Wolf-Rayet stars and are stars that lost their hydrogen envelope.*)

2. The four Very Large Telescopes can be used as an optical interferometer, with a largest baseline of 130 m. An instrument, GRAVITY, is being built specifically to observe the environment of the massive black hole in the Galactic centre (at 8 kpc), allowing one to resolve structure down to the diffraction limit.

   (a) What would be the diffraction limit if we observed at $2.2\,\mu$m (in the near-infrared K band)? Show that we could resolve a binary in the Galactic centre if its projected separation exceeded $\sim$30 AU.

   (b) Suppose we found such a just-resolved binary, with projected separation of 30 AU, and from spectra inferred it consisted of two $15\,M_\odot$ stars. Estimate the orbital period (in years). Also, calculate the minimum and maximum orbital period the binary could have.

   (c) If we measured Doppler shifts for both stars, what would be the maximum difference in velocity we might expect to measure? Make a sketch of the situation under which this would happen. (*You should sketch the (bound) orbits of the two stars, and indicate the direction to the observer.*)

3. We again consider the Galactic centre and its black hole with mass $M_{\rm BH} = 3 \times 10^6\,M_\odot$. Suppose there is a binary system at a distance r from the black hole, with the binary containing two equal-mass stars with masses $M_1 = M_2 = M_{\rm bin}/2$ that are on a circular orbit with separation s. We consider whether the binary might become unbound by the black hole's tides.

   (a) Write down the expressions for the total mutual acceleration of the stars in the binary as well as the acceleration induced by the black hole's tides. Sketch the situation. Show that the accelerations are equal when $r = s(M_{\rm BH}/M_{\rm bin})^{1/3}$. (*Assume all objects are aligned; do not worry about factors of order unity.*)

   (b) Above we assumed aligned objects. For what orientation of the binary orbit relative to the direction to the black hole would tides be much less likely to unbind the binary? Sketch this situation. Now assume the binary itself is on a parabolic orbit around the black hole. Could it remain in a safe orientation?

   (c) Use the result of (a) to estimate the closest distance to the black hole for which a binary with $M_1 = M_2 = 15\,M_\odot$ and $s = 30\,\mathrm{AU}$ (i.e., one like that in question 2) would likely remain bound. Describe what would happen to the two stars if the binary had a parabolic orbit around the black hole that brought it closer that this critical distance. Sketch the situation.

   (d) **Bonus**: If the above binary was in a circular orbit around the black hole at the critical distance, what would be the orbital period? You should find that it is similar to the binary period. Show that this follows generically from equating the mutual and tidal accelerations, as long as $M_{\rm BH} \gg M_{\rm bin}$.