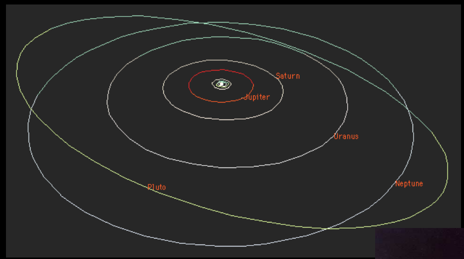
Laws of Gravity I

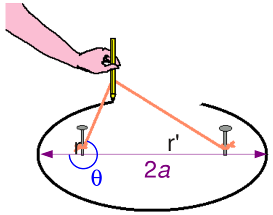


Kepler's (1571-1630) three laws (rules) for planetary motion

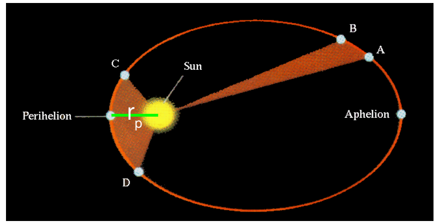
I. A planet orbits the sun in an ellipse with the sun at one focus

II. Planet-sun line sweeps out equal area in equal time intervals

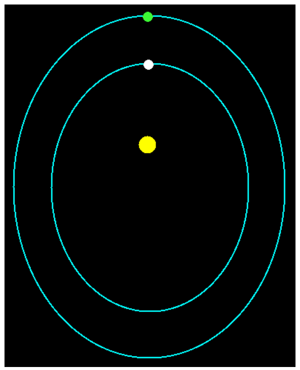
III. $P^2 = a^3$ (for the Sun, with P in years, a in AU)



$$r_p = a(1-e)$$
$$r_a = a(1+e)$$
$$A = \pi a^2 \sqrt{1-e^2}$$



Kepler III: $P^2 = a^3$

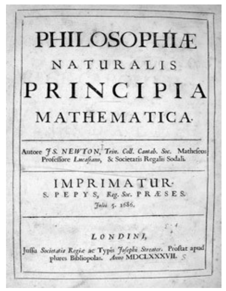


Newton's (1642-1727) three great **laws**

I. the law of inertia

II. $\mathbf{F} = m\,\mathbf{a}$

III. $\mathbf{F}_{12} = -\mathbf{F}_{21}$



Newton's law of universal gravitation
(tested broadly)

$$F = G\frac{M\,m}{r^2}$$

Now: onto deriving Kepler's laws...

Gravity only, hence conservation of
- momentum P,
- angular momentum L,
- energy E.

Linear momentum $\mathbf{P}=\sum_k \mathbf{P}_k=\sum_k m_k \mathbf{v}_k$

$$\frac{d\mathbf{P}_k}{dt}=\sum_i \mathbf{F}_{ik},\quad \mathbf{F}_{ik}=-\mathbf{F}_{ki}$$

hence, $\frac{d\mathbf{P}}{dt}=\frac{d\sum_k \mathbf{P}_k}{dt}=0$

Angular momentum $\mathbf{L}=\mathbf{r}\times\mathbf{P}$

$$\frac{d\mathbf{L}}{dt}=\frac{d\mathbf{r}}{dt}\times\mathbf{P}+\mathbf{r}\times\frac{d\mathbf{P}}{dt}=\mathbf{v}\times\mathbf{P}+\mathbf{r}\times\mathbf{F}$$

for gravity $\mathbf{F}=-G\frac{Mm}{r^2}\frac{\mathbf{r}}{r}$

hence, $\frac{d\mathbf{L}}{dt}=0$

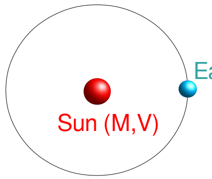
Total energy $E=E_{kin}+E_{pot}$

$$E_{kin}=\frac{1}{2}mv^2,\ E_{pot}=m\Phi=-m\frac{GM}{r}$$

$$\frac{dE_{kin}}{dt}=\mathbf{F}\cdot\mathbf{v}=-m\nabla\Phi\cdot\frac{d\mathbf{r}}{dt}=-m\frac{d\Phi}{dt}=-\frac{dE_{pot}}{dt}$$

hence, $\frac{dE}{dt}=\frac{dE_{kin}+dE_{pot}}{dt}=0$

Deriving Kepler's Laws using conservation of P, L & E



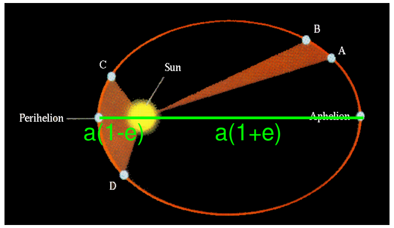
m << M *(see next lecture for m ~ M)*

P = m **v** + M **V** = constant *(= 0 if relative to center of mass)*

⇒ reflex motion of the Sun very small
So, we will ignore **V** for L and E
(but not for detecting extra-solar planets!)

L = **r** x **P** = **r** x m **v** = constant

⇒ Motion in a plane (K I partial) *(for remainder, see book)*
⇒ constant swept-out area per unit time (K II)



Conservation of P, L, E
⇒ Kepler's 3rd law

at perihelion, aphelion,

$$L=mrv=\text{constant}$$

$$E=\frac{1}{2}mv^2-\frac{GMm}{r}=\text{constant}$$

since $r_p=a(1-e),\quad r_a=a(1+e)$

$$v_p^2=\frac{GM}{a}\left(\frac{1+e}{1-e}\right)\qquad v_a^2=\frac{GM}{a}\left(\frac{1-e}{1+e}\right)$$

also, $L=m\sqrt{GMa(1-e^2)},\quad E=-\frac{GMm}{2a}$

$$\frac{dA}{dt}=\frac{L}{2m}=\text{constant}=\frac{A}{P}$$

for an ellipse, $A=\pi a^2\sqrt{1-e^2}$

hence, $P^2=\frac{4\pi^2}{GM}a^3$ *(K III)*

A few more notes on angular momentum & energy:

$E=-\frac{GMm}{2a}$; *depends only on* a, *not on* e

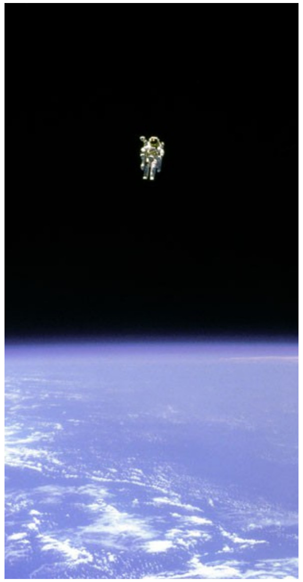
$L=m\sqrt{GMa(1-e^2)}$; *for given L, minimum E is for* $e=0$

$$E=\frac{1}{2}mv^2-\frac{GMm}{r}=-\frac{GMm}{2a}$$

$$E=E_{kin}+E_{pot}\ =\ \frac{1}{2}\langle E_{pot}\rangle=-\langle E_{kin}\rangle$$

(the Virial Theorem, we'll return to it!)

Escape velocity is when $E=0$, *which implies* $v_{esc}=\sqrt{\frac{2GM}{r}}=\sqrt{2}\,v_{circ}$



Puzzle: SPACE SURVIVAL RULE #1

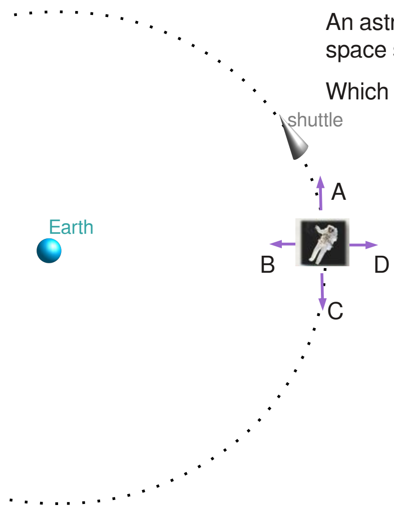
An astronaut is accidentally left behind the space shuttle by his careless colleagues.

Which way should he aim to catch up?

$$L = m\sqrt{GMa(1-e^2)}$$

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a} < 0$$

$$P^2 = \frac{4\pi^2}{GM}a^3$$



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(Tip: look up Gemini 4)