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Bounded cosmic string backreaction and connecting Green's functions for the wave equation

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Abstract

First, a brief analysis on the cosmic string equation of motion and backreaction is presented, along with simplifications through physical setup and gauge choices, for the one dimensional bounded cosmic string. This analysis follows the framework constructed by Chernoff et al. (2019), where the Green's function of the linearized Einstein equation is eventually needed.

Then, we change our focus to present a complete analysis of the wave equation under various boundary conditions: infinite string, semi-infinite string, periodic, Dirichlet, and Neumann. Through that, potential problems are considered and relationships between these boundary conditions are connected.

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Contents

1 Introduction

Cosmic superstrings are one dimensional strings from string theory. They are stretched to macroscopic scale during the early inflationary phase of the universe [1]. After their formation, these strings can interact with each other through accelerations, collisions causing breaking and reattachments, and forming loops [2]. There have been many numerical studies on their evolution and distribution on a cosmological scale [3, 4]. In addition, there have been work to place experimental constraints on these strings using various astrophysical techniques ranging from lensing to gravitational wave background [5–7].

With all these properties, evolution, and associated phenomena, cosmic superstrings are interesting objects for further analyses and studies. In this work, an approach to finding bounded cosmic string backreaction is discussed. In that approach, the Green's function for Einstein's field equation is required. This work then transitions into a careful analysis of the relationship between Green's functions for the wave equation in one dimension under various boundary conditions. This analysis establishes a method for connecting different solutions of different boundary conditions. A generalization of this technique can then potentially be used on Green's function for the Einstein's field equation, and thereby finding the backreaction of a cosmic string.

The first section establishes the theoretical framework and background of cosmic strings: its equation of motion, and the formalism to finding the backreaction. In the subsequent sections, the Green's functions for the one dimensional wave equation under the following conditions are analyzed and connected: infinte string, semi-infinite string, periodic, Dirichlet, and Neumann. Then, a discussion of possible generalizations to higher dimensions is presented, as a bridge to characterizing

Green's function for the Einstein field equation under various boundary conditions.

1.1 Cosmic String Setup

First, let us consider a Nambu-Goto string. This string exists on some background metric $g_{\alpha\beta}$. It also traces out a two dimensional worldsheet γ_{ab}, parametrized by the worldsheet coordinates $\zeta^a = (\tau, \sigma)$. The first parameter τ can be chosen to be timelike, and subsequently σ to be spacelike. In a way, σ identifies points on a string. Therefore, we can also parametrize the spacetime coordinate by ζ^a:

$$z^\alpha = z^\alpha(\zeta^a). \tag{1.1}$$

Note that Latin indices range from 0 to 1, whereas Greek indices range from 0 to 4. Furthermore, we can project the spacetime metric $g_{\alpha\beta}$ onto the worldsheet such that:

$$\gamma_{ab} = g_{\alpha\beta}\partial_a z^\alpha \partial_b z^\beta. \tag{1.2}$$

Now, in the low string tension regime, the spacetime metric can be treated perturbatively [2], where the string is on a metric defined as:

$$g_{\alpha\beta} = \mathring{g}_{\alpha\beta} + h_{\alpha\beta} \tag{1.3}$$

where $\mathring{g}_{\alpha\beta}$ the background metric, and $h_{\alpha\beta}$ the metric perturbation induced by the string's backreation. Finding $h_{\alpha\beta}$ is the goal. In a similar way, we can perturbatively parametrize the worldsheet:

$$z^\alpha = z^\alpha_{(0)} + z^\alpha_{(1)}. \tag{1.4}$$

1.2 Simplifications, Gauge Choice, and Equation of Motion

The equation of motion for a string without any backreaction perturbation is [2]:

$$\frac{1}{\sqrt{-\gamma}}\partial_a(\sqrt{-\gamma}\gamma^{ab}\partial_b z^{\gamma}_{(0)}) - P^{\alpha\beta}\Gamma^{\gamma}_{\alpha\beta} = 0 \tag{1.5}$$

where $\gamma = \det \gamma_{ab}$. Now, we can make our first simplification by letting the string live in a flat background spacetime. This way, we only analyze the backreaction itself. With that, the second term vanishes since the Christoffel symbol is zero in a flat spacetime. At this point, the equation of motion becomes

$$\partial_a(\gamma^{ab}\partial_b z^{\gamma}) = 0. \tag{1.6}$$

We can also choose the conformal gauge on the worldsheet by requiring that[8]:

$$\begin{aligned} \gamma_{00} + \gamma_{11} &= 0 \\ \gamma_{01} &= 0. \end{aligned} \tag{1.7}$$

One implication choosing the conformal gauge is that the worldsheet becomes conformally flat:

$$\gamma_{ab} = \sqrt{-\gamma}\eta_{ab} \tag{1.8}$$

where η_{ab} the flat Minkowski metric. Furthermore, using equation 1.2 with the conformal gauge, we acquire the following constraints:

$$\begin{aligned} \partial_\tau z \cdot \partial_\sigma z &= 0 \\ (\partial_\tau z)^2 + (\partial_\sigma z)^2 &= 1. \end{aligned} \tag{1.9}$$

In addition, the string equation of motion takes the form:

$$\partial_\tau^2 z^{\gamma} - \partial_\sigma^2 z^{\gamma} = 0. \tag{1.10}$$

Note that this is simply the 2D wave equation. At this point, we make our simplification and choose to suppress the y dimension such that the string now oscillates in only one dimension. That is, set $z^2 = y = 0$. There is a residual gauge choice [8], which we will define:

$$\begin{aligned} z^0 &= t \equiv \tau \\ z^1 &= z \equiv \sigma. \end{aligned} \tag{1.11}$$

With these simplifications and gauge choices, we now have a wave equation for $z^1 = x$ in terms of τ and σ, and the direction of propagation is along $z^3 = z$. Therefore, our task is now simply to find $x(\tau, \sigma)$ obeying the wave equation. Now, we need to set the boundary condition for this problem. We choose to set the hardwall (Dirichlet) boundary condition for a string of finite length L:

$$x(\sigma = 0) = x(\sigma = L) = 0. \tag{1.12}$$

Now, we have the solution for the wave equation under this boundary condition:

$$\begin{aligned} x(\tau, \sigma) = \frac{1}{2} \sum_n^{\infty} &\left[-A_n \sin\left(\frac{n\pi}{L}(\sigma - \tau)\right) + B_n \cos\left(\frac{n\pi}{L}(\sigma - \tau)\right)\right] + \\ &\left[A_n \sin\left(\frac{n\pi}{L}(\tau + \sigma)\right) - B_n \cos\left(\frac{n\pi}{L}(\tau + \sigma)\right).\right] \end{aligned} \tag{1.13}$$

It is important to note that this solution is obviously equivalent to the D'Alembert solution of the form:

$$x(\tau, \sigma) = \frac{1}{2}(\mathcal{F}(\sigma - \tau) + \mathcal{G}(\sigma + \tau)). \tag{1.14}$$

and the constraints we have from the gauge choices give us the following constraint on $\mathcal{F}, \mathcal{G}$:

$$(\partial_\tau \mathcal{F})^2 = (\partial_\tau \mathcal{G})^2 = 1. \tag{1.15}$$

Thus, with suitable simplifications and gauge choices, we have completely described the unperturbed equation of motion. The next step is finding out the perturbations.

1.3 Green's Function and Backreaction

At this point, we continue to follow the formalism laid out in [2]. The next step is finding the retarded (causal) Green's function satisfying the linearized Einstein field equation with a source at x' (in spacetime) [2]:

$$\Box G_{\alpha\beta}{}^{\alpha'\beta'} + 2R_{\alpha}{}^{\gamma}{}_{\beta}{}^{\delta} G_{\gamma\delta}{}^{\alpha'\beta'} = -g_{\alpha}{}^{\alpha'} g_{\beta}{}^{\beta'} \delta^4(x, x') \tag{1.16}$$

where $R_{\alpha}{}^{\gamma}{}_{\beta}{}^{\delta}$ is the Riemann tensor. Note that this is the linearized Einstein's equation, after choosing the Lorenz gauge. Then, the perturbation metric $h_{\alpha\beta}$ is retrieved by a convolution of the Green's function with a source (the stress energy tensor $T_{\alpha'\beta'}$ to get the backreation because the backreaction is caused by the stress-energy of the string itself), giving:

$$\bar{h}_{\alpha\beta}(x) = 16\pi \int G^{(\text{ret})\alpha'\beta'}_{\alpha\beta}(x.x') T_{\alpha'\beta'}(x') \sqrt{-g(x')} d^4x' \tag{1.17}$$

where $\bar{h}_{\alpha\beta}(x) = h_{\alpha\beta} - \frac{1}{2}\mathring{g}_{\alpha\beta}\mathring{g}^{\gamma\delta}h_{\gamma\delta}$. From [2], this is equivalent to:

$$\bar{h}_{\alpha\beta}(x) = -4G\mu \int \int P_{\alpha\beta}\delta[\sigma(x,z)]\sqrt{-\gamma} d\tau' d\sigma' \tag{1.18}$$

where $\sigma(x, z)$ the Synge world-function, μ the string tension, and $P_{\alpha\beta}$ the world tangent projection operator defined as $P^{\alpha\beta} = \gamma^{ab}\partial_a z^\alpha \partial_b z^\beta$. In fact, with the equation of motion we found from the previous section:

$$P^{\alpha\beta} = \begin{bmatrix} -1 & \partial_\tau x & 0 & -1 \\ -\partial_\tau x & -(\partial_\tau x)^2 + (\partial_\sigma x)^2 & 0 & \partial_\sigma x \\ 0 & 0 & 0 & 0 \\ -1 & \partial_\sigma x & 0 & 1 \end{bmatrix}. \tag{1.19}$$

Before we move on to discuss about Green's functions, a brief discussion on the backreaction is important. These cosmic strings radiate gravitational waves due to their stress-energy tensor. As a result, they also experience a backreaction force, similar to the electron backreaction typically encountered in Electromagnetism [9, 10]. In the string context, the backreaction induces a perturbation on the background metric itself, affecting the string's equation of motion. It is this perturbation of the background metric and the backreaction force that we want to find. Furthermore, in this setup we bound the string between two hardwall. The motivation for that is once we understand the backreaction, we can analyze the radiation output to the walls as a function of time. With this, we can find the energy decay of the string over some characteristic time. This is crucial for the understanding of cosmic string properties over the cosmological timescale.

Now, we notice that to find the backreaction perturbations to the spacetime metric, we must find the Green's function for the linearized Einstein field equation. At this point, we change our focus onto the Green's function itself. In [2], the Green's function for the linearized Einstein field equation is considered for the closed loop case with periodic boundary condition. Here, we are trying to consider a finite string stretched between two walls. Some natural questions arise: (1) Can we move from one solution under a certain boundary condition to another? and (2) What is the relationship between various boundary conditions?

In the following sections, we wish to analyze the Green's function for the wave equation for various boundary conditions and attempt to connect these solutions. The wave equation is not the linearized Einstein field equation that we wanted to solve, but it is very similar in form. In fact, the linearized Equation on flat spacetime ($R_{\alpha\ \beta}^{\ \gamma\ \delta} = 0$) gives us the wave equation. Thus, we speculate that this analysis for the wave equation is transferable to the Einstein equation. A clear advantage of analyzing the wave equation is its well-known solution forms and theorems (such as uniqueness). In addition, we restrict this analysis mostly in 1D, but it is easily extendable to higher dimensions. Finally, it is important to note that here we are analyzing the evolution of these systems. This can be best explained through an analogy with Electromagnetism. The first two Maxwell's equations constrain the physics of the system, whereas the latter two govern its evolution. We are interested in the evolution when we analyze the Green's function.

2 Infinite String with Source

Consider the wave equation with a single (Dirac-delta) source at $(t, \mathbf{r}) = (t_0, \mathbf{r}_0)$ in D dimension:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u(\mathbf{r}, t) = \delta(t - t_0)\ \delta^{(D)}(\mathbf{r} - \mathbf{r}_0) \tag{2.1}$$

where δ^D the generalized Dirac-delta in D dimension, and c the propagation velocity. We seek to find the Green function u such that the boundary condition $u(\mathbf{r} = \pm\infty, t) = 0$ is satisfied. Furthermore, there is no signal before t_0 (causality condition); that is, $u(\mathbf{r}, t) = 0, \partial_t u(\mathbf{r}, t) = 0$ for all $t < t_0$.

We can perform a simple coordinate change to simplify the problem:

$$\mathbf{R} := \mathbf{r} - \mathbf{r}_0,\ \ \tau := t - t_0. \tag{2.2}$$

The problem we are solving now becomes:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} - \nabla^2_{\mathbf{R}}\right) u(\mathbf{R}, \tau) = \delta(\tau)\ \delta^{(D)}(\mathbf{R}). \tag{2.3}$$

Now the boundary condition becomes $u(\mathbf{R} = \pm\infty, \tau) = 0$ and the causality condition becomes $u(\mathbf{R}, \tau) = 0, \partial_\tau u(\mathbf{R}, \tau) = 0$ for all $\tau < 0$. In Figure 1, we observe that this causality condition gives us the causal Green's function, emitting into the future from the source at $x = x_0$ and $\tau = 0$. The advanced Green's function (anti-causal) lightcone is also depicted as dashed lines.

2.1 Fourier Transform

The (forward) Fourier transform for the pair $\tau - \omega$ and $\mathbf{R} - \mathbf{k}$ is:

$$\tilde{u}(\mathbf{k}, \omega) = \int \mathrm{d}^{(D)}\mathbf{R} \int \mathrm{d}\tau e^{-i(\mathbf{k}\cdot\mathbf{R} - \omega\tau)}\ u(\mathbf{R}, \tau) \tag{2.4}$$

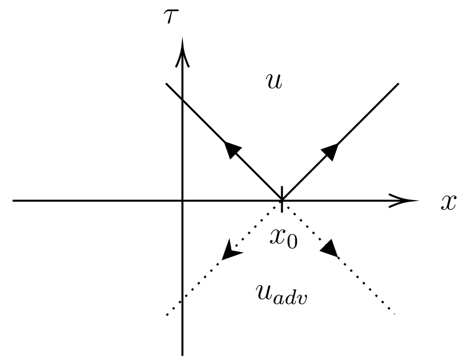

Figure 1: The causal Green's function u and the advanced Green's function u_{adv} (dashed line) shown as lightcone emitting from the source at $x = x_0, \tau = 0$.

and the inverse Fourier transform is:

$$u(\mathbf{R}, \tau) = \int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D} \int \frac{\mathrm{d}\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{R} - \omega\tau)} \; \tilde{u}(\mathbf{k}, \omega). \tag{2.5}$$

Using these definitions and the properties of the δ function, we can write:

$$\delta^{(D)}(\mathbf{R})\delta(\tau) = \int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D} \int \frac{\mathrm{d}\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{R} - \omega\tau)}. \tag{2.6}$$

Using equations 2.5 and 2.6 in 2.3, we get:

$$\begin{gathered} \int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D} \int \frac{\mathrm{d}\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{R} - \omega\tau)} (\omega^2 - c^2 k^2)\tilde{u} = -c^2 \int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D} \int \frac{\mathrm{d}\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{R} - \omega\tau)} \\ \int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D} \int \frac{\mathrm{d}\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{R} - \omega\tau)} \; [(\omega^2 - c^2 k^2)\tilde{u} + c^2] = 0. \end{gathered} \tag{2.7}$$

This can only vanish if $(\omega^2 - c^2k^2)\tilde{u} + c^2 = 0$. Note that here $k = |\mathbf{k}|$.

$$\implies \tilde{u}(\mathbf{k}, \omega) = -\frac{c^2}{\omega^2 - c^2 k^2} \tag{2.8}$$

Using the Fourier transform (equation 2.5), we can get the desired $u(\mathbf{R}, \tau)$:

$$u(\mathbf{R}, \tau) = -c^2 \int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D} e^{i\mathbf{k}\cdot\mathbf{R}} \int \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega\tau} \left(\frac{1}{\omega^2 - c^2 k^2} \right). \tag{2.9}$$

The next task is evaluating this integral.

2.2 The ω integral

Lemma 2.1. (Jordan's Lemma)[11]*: If $m > 0$ and $\frac{P}{Q}$ is the quotient of two polynomials such that:*

$$\text{degree}(Q) \geq 1 + \text{degree}(P)$$

then:

$$\lim_{R \longrightarrow \infty} \int_{C_{R^+}} e^{imz} \frac{P(z)}{Q(z)} dz = 0$$

where C_{R^+} is the upper half-circle with radius R.

<u>Remark</u>*: if $m < 0$, then*

$$\lim_{R \longrightarrow \infty} \int_{C_{R^-}} e^{imz} \frac{P(z)}{Q(z)} dz = 0$$

where C_{R^-} the lower half-circle with radius R.

First, consider the integral involving ω. Observe that there are two simple poles of order 1 at $\omega = \pm ck$. This integral can be done in ω-space, extending $\omega \in \mathbb{C}$, and use the residue theorem from complex analysis.

First, let $\mathbb{R} \ni R > c|k|$. Consider the contour, Γ^+ consisting of a line from $-R$ to R and a semicircle C^+ of radius R on the upper half plane connecting $-R$ and R. With this, we have:

$$\left(\int_{-R}^{R} + \int_{C^+} \right) \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega\tau} \left(\frac{1}{\omega^2 - c^2k^2} \right). \tag{2.10}$$

If $\tau < 0$, then the semicircle lies on the upper half-plane of ω because then its' contribution to the integral is zero as $R \longrightarrow \infty$ because of Jordan's Lemma. Thus, if there are no poles on the the real line, then this integral is zero due to Cauchy's Integral Theorem. This is exactly what we want because of the causality condition that $u(\mathbf{R}, \tau) = 0$ when $\tau < 0$.

Now, if $\tau > 0$, then the semicircle is on the lower half-plane (see Remark of Jordan's Lemma). Furthermore, we expect that if $\tau > 0$, then $u(\mathbf{R}, \tau)$ does not vanish for all $(\mathbf{R}, \tau)$. The only way for this condition to satisfy is if the two singularities $\omega = \pm ck$ are in the lower half-plane.

We can shift the two poles to the lower half-plane by a factor of $-i\epsilon$, $\epsilon \in \mathbb{R}$. That is, we now have poles at $\omega = \pm ck - i\epsilon$. After the integral evaluation, we then take limit $\epsilon \longrightarrow 0$ to recover our result.

Now, we have the contour Γ_- as shown in Figure 2. Furthermore, the integral itself becomes:

$$\begin{aligned} &\lim_{\epsilon \longrightarrow 0} \oint_{\Gamma_-} \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega\tau} \left(\frac{1}{(\omega + ck + i\epsilon)(\omega - ck - i\epsilon)} \right) = \\ &\lim_{\epsilon \longrightarrow 0} -2\pi i(\mathrm{Res}(-ck - i\epsilon) + \mathrm{Res}(ck - i\epsilon)) \end{aligned} \tag{2.11}$$

where the Cauchy's Residue Theorem is used in the second step. Evaluating the residue of these two poles:

$$\begin{aligned} \mathrm{Res}(ck - i\epsilon) &= \lim_{w \longrightarrow ck - i\epsilon} \frac{e^{-i\omega\tau}}{w + ck + i\epsilon} = \frac{e^{-i(ck - i\epsilon)\tau}}{2ck} \\ \mathrm{Res}(-ck - i\epsilon) &= \lim_{w \longrightarrow -ck - i\epsilon} \frac{e^{-i\omega\tau}}{w - ck - i\epsilon} = \frac{e^{i(ck + i\epsilon)\tau}}{2ck}. \end{aligned} \tag{2.12}$$

Combining the residue and evaluate the ϵ limit, we have the solution to the ω integral:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{e^{-i\omega\tau}}{\omega^2 - c^2k^2} = -\frac{\sin(c\tau k)}{ck}. \tag{2.13}$$

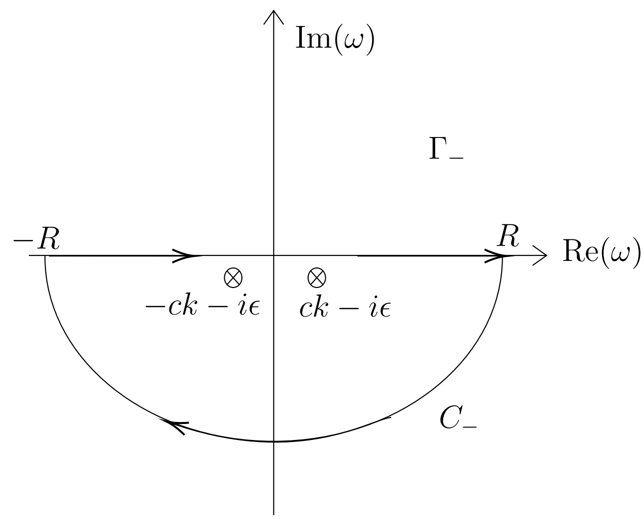

Figure 2: The contour Γ_- with the two poles in the lower half-plane of ω.

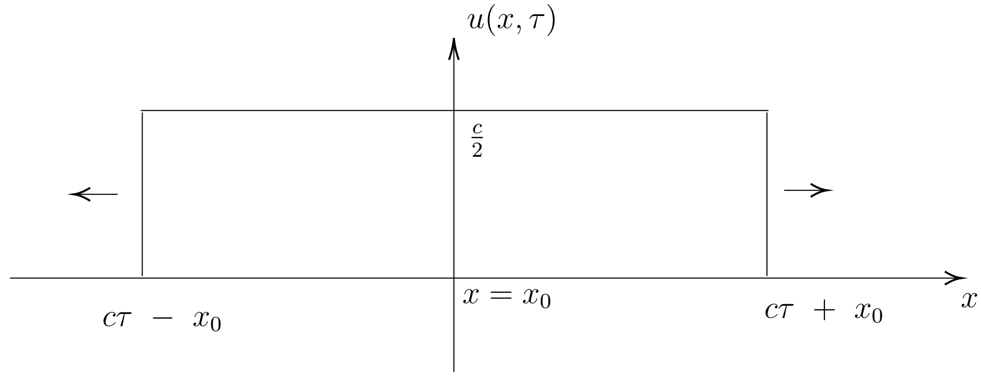

Figure 3: The 1D Green's function behavior.

The solution to our wave equation is now:

$$u(\mathbf{R}, \tau) = c\Theta(\tau) \int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D} \; e^{i\mathbf{k}\cdot\mathbf{R}} \; \frac{\sin(c\tau k)}{k} \tag{2.14}$$

where we introduced the Heaviside function $\Theta(\tau)$ to strictly enforce the causality condition that u vanishes when $\tau < 0$.

2.3 The k integral in 1D

In 1D, equation 2.14 becomes:

$$\begin{aligned} u_1(R,\tau) &= c\Theta(\tau)\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\ e^{ikR}\ \frac{\sin(c\tau k)}{k} \\ &= c\Theta(\tau)\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\ \frac{\cos(kR)\sin(c\tau k)}{k} + ic\Theta(\tau)\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\ \frac{\sin(kR)\sin(c\tau k)}{k} \\ &= c\Theta(\tau)\int_{0}^{\infty}\frac{\mathrm{d}k}{2\pi}\ \frac{\sin(k(c\tau+R))+\sin(k(c\tau-R))}{k}. \end{aligned} \tag{2.15}$$

Now, use the known result:

$$\int_0^{\infty}\mathrm{d}x\ \frac{\sin(ax)}{x} = \frac{\pi}{2}\mathrm{sgn}(a) \tag{2.16}$$

where sgn is the sign function. With this we have the full solution for the 1D case:

$$u_1(R,\tau) = \frac{c\Theta(\tau)}{4}\left(\mathrm{sgn}(c\tau+R)+\mathrm{sgn}(c\tau-R)\right) = \frac{c}{2}\Theta(\tau)\Theta(c\tau-|R|) \tag{2.17}$$

where $R = x - x_0$ and $\tau = t - t_0$. In Figure 3, we observe the behavior of the Green's function $u(x,\tau)$: the Green's function is a constant, non-damping emitting signal from $x = x_0, \tau = 0$. As τ increases, the Green's function expands and the signal reaches more field points. This expansion is equivalent to the lightcone that we have seen in Figure 1.

3 Closed String

Again, we consider the wave equation with a single source. This time, let us consider the case in one dimension, with a string of length L

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u(x,t) = \delta(t-t_0)\ \delta(x-x_0). \tag{3.1}$$

Now, to create a 1-D closed string, we work in $D = 1$ with a string of length L

$$\begin{gathered} u = u(x,t) \\ 0 < x, x_0 < L; \ \ -\infty < t, t_0 < \infty \end{gathered} \tag{3.2}$$

and impose a periodic boundary condition:

$$u(0,t) = u(L,t),\ \partial_x u(0,x) = \partial_x u(L,t). \tag{3.3}$$

Similar to before, define $\tau := t - t_0$. We also impose the causality condition:

$$u(x,\tau) = 0,\ \partial_t u(x,\tau) = 0,\ \tau < 0. \tag{3.4}$$

3.1 Integrating the Wave Equation

3.1.1 The Spatial Part

Before we move on to solving this problem, there is an important issue we must resolve first. Consider the spatial part of this problem:

$$-\frac{\partial^2 u}{\partial x^2} = \delta(x-x_0) \tag{3.5}$$

where $u = u(x); 0 < x, x_0 < L$ and we have the periodic boundary condition:

$$u(0) = u(L),\ \partial_x u(0) = \partial_x u(L). \tag{3.6}$$

Integrating both sides of the equation 3.5 over the domain $x \in [0, L]$, we get:

$$-\int_0^L \frac{\partial^2 u}{\partial x^2} = \int_0^L \delta(x - x_0). \tag{3.7}$$

The right hand side is 1 because $0 < x, x_0 < L$ and we can use the Fundamental Theorem of Calculus to evaluate left hand side:

$$-\left(\partial_x u(L) - \partial_x(0)\right) = 1. \tag{3.8}$$

However, this is a contradiction because $\partial_x(L) - \partial_x(0) = 0$. Thus, this problem needs to be re-posed on the right hand side to ensure that it is compatible with the periodic boundary condition:

$$-\frac{\partial^2 u}{\partial x^2} = \delta(x - x_0) - 1. \tag{3.9}$$

That is, we must introduce a constant equals to the sum of the sources introduced.

3.1.2 The Wave Equation

Now, we need to check if the wave equation encounters this problem. To recap, we have the following setup:

$$\begin{gathered}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u(x,t) = \delta(x - x_0)\delta(t - t_0) \\ 0 < x, x_0 < L;\ 0 < t, t_0 < \infty \\ u(0,t) = u(L,t);\ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}.\end{gathered} \tag{3.10}$$

We can make the transformation $\tau := t - t_0$ and enforce the causality condition that $u(x,\tau) = 0$ when $\tau < 0$. Now, integrate both sides of the wave equation for $x \in [0, L]$ and $\tau \in [-\epsilon, \epsilon]$, $\epsilon > 0$, which is the domain $[0, L] \times [-\epsilon, \epsilon]$:

$$\int_0^L \mathrm{d}x \int_{-\epsilon}^{\epsilon} \mathrm{d}\tau \frac{1}{c^2}\frac{\partial^2 u}{\partial \tau^2} - \int_0^L \mathrm{d}x \int_{-\epsilon}^{\epsilon} \mathrm{d}\tau \frac{\partial^2 u}{\partial x^2} = \int_0^L \mathrm{d}x \int_{-\epsilon}^{\epsilon} \mathrm{d}\tau \delta(\tau)\delta(x - x_0). \quad (3.11)$$

We can easily see that the right hand side is 1. On the left hand side, we can perform the second term first. The domain $[0, L] \times [-\epsilon, \epsilon]$ allows us to use Fubini's Theorem to change the order of integration:

$$\int_0^L \mathrm{d}x \int_{-\epsilon}^{\epsilon} \mathrm{d}\tau \frac{1}{c^2}\frac{\partial^2 u}{\partial \tau^2} - \int_{-\epsilon}^{\epsilon} \mathrm{d}\tau \int_0^L \mathrm{d}x \frac{\partial^2 u}{\partial x^2} = 1. \quad (3.12)$$

We can now evaluate the x integral of the second term:

$$\int_0^L \mathrm{d}x \int_{-\epsilon}^{\epsilon} \mathrm{d}\tau \frac{1}{c^2}\frac{\partial^2 u}{\partial \tau^2} - \int_{-\epsilon}^{\epsilon} \mathrm{d}\tau \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) = 1. \quad (3.13)$$

Now, use the periodic boundary condition to see that $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} = 0$ and we have:

$$\int_0^L \mathrm{d}x \int_{-\epsilon}^{\epsilon} \mathrm{d}\tau \frac{1}{c^2}\frac{\partial^2 u}{\partial \tau^2} = 1. \quad (3.14)$$

Next, we can perform the τ integral and use the causality condition $u(x,\tau) = 0$ for $\tau < 0$:

$$\begin{aligned} \int_0^L \mathrm{d}x \frac{1}{c^2}\left(\frac{\partial u}{\partial \tau} - \frac{\partial u}{\partial \tau}\right) &= 1 \\ \int_0^L \mathrm{d}x \frac{1}{c^2}\frac{\partial u}{\partial \tau} &= 1 \\ \frac{1}{c^2}\lim_{\tau \longrightarrow \epsilon} \int_0^L \mathrm{d}x \frac{\partial u}{\partial \tau} &= 1. \end{aligned} \quad (3.15)$$

Using Leibniz's rule to move the partial derivative outside:

$$\frac{1}{c^2}\lim_{\tau \longrightarrow \epsilon} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^L \mathrm{d}x \; u(x,\tau) = 1 \quad (3.16)$$

where ϵ is just any $\tau > 0$ This condition must always be satisfied at any $\tau > 0$. Clearly, the wave equation does not face the same problem that the spatial equation with source does and we can move forward to solving the problem without having to add a correction term to the source.

3.2 Temporal Fourier Transform

Now, we can start solving. First, take the temporal Fourier transform $(\omega - \tau)$ to transform this into the Hemholtz equation with periodic boundary conditions:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\omega^2}{c^2}\right)\tilde{u}(x,\omega) = \delta(x - x_0) \tag{3.17}$$

where we have the Fourier transformed

$$\tilde{u}(x,\omega) = \int \frac{\mathrm{d}\omega}{2\pi} e^{i\omega\tau} u(x,\tau). \tag{3.18}$$

Note that equation 3.17 is the Hemholtz equation with the same periodic boundary condition as before. Our task now is to find $\tilde{u}$, then perform an inverse Fourier transform to get the desired Green function u.

3.3 Bilinear Expansion

Lemma 3.1. (Green's Function Bilinear Expansion)[12] *: Given a differential operator L with a set of complete eigenvectors $\phi_n(x)$ and corresponding eigenvalues λ_n, then the Green's function satisfying L can be constructed by:*

$$G(x, x_0) = \sum_n \frac{\phi_n(x)\phi_n^*(x_0)}{\lambda_n - \lambda}.$$

The approach here is using Green's function bilinear expansion. To do this, use the dispersion relation $k = \frac{\omega}{c}$ and define the Sturm-Liouville operator $L = \frac{\partial^2}{\partial x^2}$.

With this 3.17 becomes:

$$(L - k^2)\tilde{u}(x, \omega) = \delta(x - x_0). \tag{3.19}$$

The solution to $\tilde{u}$ can be constructed from eigenfunctions ϕ_n and eigenvalues λ_n using the bilinear expansion:

$$\tilde{u}(x, \lambda) = \sum_n \frac{\phi_n(x)\phi_n^*(x_0)}{\lambda_n - \lambda}. \tag{3.20}$$

ϕ_n and with its eigenvalue λ_n are found by solving the homogeneous equation:

$$(L - \lambda_n)\phi_n(x) = 0. \tag{3.21}$$

Comparing equations 3.19 and 3.21, we see that $\lambda_n = k_n^2$. Since $L = \frac{\partial^2}{\partial x^2}$, this is easy to solve with solutions

$$\phi_n = A_n e^{i\sqrt{\lambda_n}x} + B_n e^{-i\sqrt{\lambda_n}x},\ n = 0, 1, 2, ... \tag{3.22}$$

Using the periodic boundary conditions, we find that:

$$\begin{aligned} A + B &= Ae^{i\sqrt{\lambda}x} + Be^{-i\sqrt{\lambda}x} \\ A - B &= Ae^{i\sqrt{\lambda}x} - Be^{-i\sqrt{\lambda}x} \\ \implies \sqrt{\lambda_n} &= \frac{2n\pi}{L}, \implies \phi_n = C_n e^{i\sqrt{\lambda_n}x},\ n \in \mathbf{Z}. \end{aligned} \tag{3.23}$$

Since $\sqrt{\lambda_n} = k_n$ and $C_n = \sqrt{\frac{1}{L}}$ by normalization, we have the following eigenfunctions:

$$\phi_n = \sqrt{\frac{1}{L}} e^{ik_n x},\ n \in \mathbf{Z}. \tag{3.24}$$

3.3.1 Completeness Relation

It is important to check that our eigenfunctions ϕ_n satisfy the completeness relation:

$$\sum_n \phi_n(x)\phi_n^*(x_0) = \delta(x - x_0). \tag{3.25}$$

Using ϕ_n from above:

$$\sum_{n=-\infty}^{\infty} e^{ikx}e^{-ikx_0} = \sum_{n=-\infty}^{\infty} e^{ik(x-x_0)} = \delta(x - x_0). \tag{3.26}$$

Thus, ϕ_n is a valid set of eigenfunction satisfying the boundary condition and the bilinear expansion completeness criterion.

3.3.2 The Solution

Returning to the problem, we can use ϕ_n and the dispersion relation $k = \frac{\omega}{c}$ to express $\tilde{u}$ in terms of x and ω again with the bilinear expansion from equation 3.20:

$$\tilde{u}(x,\omega) = \sum_{n\in\mathbb{Z}} \frac{1}{L}\frac{e^{ik_n(x-x_0)}}{k_n^2 - \frac{\omega^2}{c^2}} = -\frac{c^2}{L}\sum_{n\in\mathbb{Z}}\frac{e^{ik_n(x-x_0)}}{\omega^2 - c^2k_n^2}. \tag{3.27}$$

Now, we can perform the inverse Fourier transform to get the Green function $u(x,\tau)$:

$$\begin{aligned} u(x,\tau) &= -\frac{c^2}{L}\int \frac{\mathrm{d}\omega}{2\pi}e^{-i\omega\tau}\sum_{n\in\mathbb{Z}}\frac{e^{ik_n(x-x_0)}}{\omega^2 - c^2k_n^2} \\ u(x,\tau) &= -\frac{c^2}{L}\sum_{n\in\mathbb{Z}}e^{ik_n(x-x_0)}\int\frac{\mathrm{d}\omega}{2\pi}e^{-i\omega\tau}\left(\frac{1}{\omega^2 - c^2k_n^2}\right). \end{aligned} \tag{3.28}$$

Note that the integral inside is the same as that in section 2.2. Using the result from the ω integral and put in $\Theta(\tau)$ to strictly enforce the causality condition, our Green function is:

$$u(x,\tau) = \frac{c}{L}\Theta(\tau)\sum_{n=-\infty}^{\infty} e^{ik_n(x-x_0)}\frac{\sin(c\tau k_n)}{k_n},\ k_n = \frac{2n\pi}{L}. \tag{3.29}$$

4 Image Sources and Eigenvalues

4.1 Infinite String, Infinite Sources

Here, we want to solve the infinte string problem, but with infinite sources, spaced by a distance Λ, as shown the lightcones of Figure 4, in one dimension:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u(x,t) = \delta(t-t_0) \sum_{n=-\infty}^{\infty} \delta(x - x_0 - n\Lambda). \tag{4.1}$$

Define the operator $L := \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ and let $R_n := x - x_0 - n\Lambda$, $\tau := t - t_0$ and the problem becomes:

$$Lu(x,\tau) = \delta(\tau) \sum_{n=-\infty}^{\infty} \delta(R_n). \tag{4.2}$$

The right hand side is a sum of single sources. Since L is a linear differential operator, we can use superposition of solutions to use the solution from the single source problem and construct the particular solution. From equation 2.14, the solution for a source at R_n is

$$u_n(x,\tau) = c\Theta(\tau) \int \frac{\mathrm{d}k}{2\pi} e^{ikR_n} \frac{\sin(c\tau k)}{k}. \tag{4.3}$$

The solution for infinite sources is the superposition of all these solutions:

$$u_n(x,\tau) = \sum_n u_n(x,\tau) = c\Theta(\tau) \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}k}{2\pi} e^{ikR_n} \frac{\sin(c\tau k)}{k}. \tag{4.4}$$

Using the result for the k-integral from the previous section, the final solution is:

$$u_n(x,\tau) = \frac{c}{2}\Theta(\tau) \sum_{n=-\infty}^{\infty} \Theta(c\tau - |x - x_0 - n\Lambda|). \tag{4.5}$$

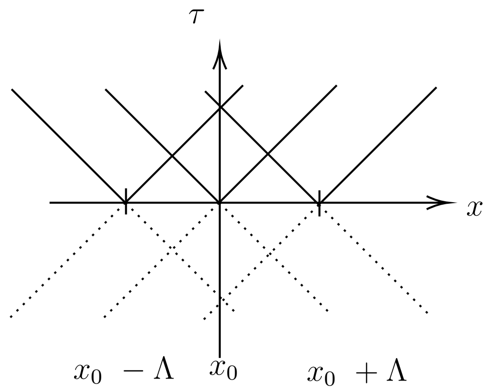

Figure 4: Lightcones emitted by sources spaced by Λ. The causal Green's function lightcone is the solid line, and the anti-causal Green's function the dashed line. There are infinitely many of these sources.

4.2 Connecting the solutions: Eigenvalues

4.2.1 Closed String to Infinite String

Before we start connecting the solution of the infinite string-infinite sources problem to the closed string problem, we must carefully analyze the closed string case first. First, we can see how to make the closed string into the infinite string.

The central claim here is: the closed string case, in the $L \to \infty$ limit (where L is the string length) gives us the infinite string case with a single source. The solution of the infinite string in equation 2.17 is periodic in the domain $-\infty < x < \infty$, another clear indication that taking this limit is valid.

Furthermore, note that the closed string case has discrete eigenstates. This can be seen because $k_n = \frac{2n\pi}{L}$ for $n \in \mathbb{Z}$. Again, if we take the limit $L \to \infty$, then we have a continuous eigenstates. Because of this, the sum in equation 3.29 becomes an integral and we have:

$$u(x,\tau) = c\Theta(\tau) \int \mathrm{d}k \; e^{ik(x-x_0)} \frac{\sin(c\tau k)}{k}. \tag{4.6}$$

which is clearly the same as the infinite string with one source case (c.f. equation 2.14).

Note that the normalization factor $\frac{1}{L}$ is removed, since we have an infinite string and normalization is no longer necessary. In addition, we are also missing a $\frac{1}{2\pi}$ factor from equation 2.14. That being said, the two forms are indeed the same and the solution forms are equivalent up to a multiplicative constant.

4.2.2 Infinite Sources to Closed String

Notice that equation 3.29 for the closed string, and equation 4.4 share remarkable similarities. This has been explored in the previous section using discrete and continuous eigenvalues. The difference here is that we are going from the continuous case back to discrete, over a finite length. Now, we have degenerate modes that are unnecessary. For the finite case, we used the bilinear expansion:

$$G(x, x_0) = \sum_n \frac{\phi_n(x)\phi_n^*(x_0)}{\lambda_n - \lambda}. \tag{4.7}$$

In the continuous eigenvalues case, this becomes [12]:

$$\lim_{\Delta\lambda_n \to 0} G(x, x_0) = \int \frac{\mathrm{d}\lambda_n}{\lambda_n - \lambda} \sum_\alpha \phi_n(x)^{(\alpha)} \phi_n^*(x_0)^{(\alpha)}. \tag{4.8}$$

where α are degenerate modes. Comparing these two forms to what we have for equations 3.29 and 4.4, it becomes obvious how they are connected. One approach is finding all these degenerate modes and constructing the solution for the closed string. However, an easier method is using the method of image charges typically used in electrostatic problems.

The uniqueness theorem [13, 14] along with the superposition principle guarantee that we can use the standard technique of image charges. That is, if our constructed solution of the infinite string-infinite sources satisfy the boundary condition, then it *must* be the solution. Thus, we want to make sure that equation 4.5 satisfies the following periodic boundary condition:

$$\begin{aligned} u(0,\tau) = u(L,\tau);\ \partial_\tau u(0,\tau) = \partial_\tau u(L,\tau) \\ (x,\tau) \in [0,L] \times [0,\infty). \end{aligned} \tag{4.9}$$

Upon inspection of equation 4.5, we can see that it is periodic over length Λ. Since it is constructed using the infinite string, we have the following periodic boundary condition:

$$\begin{aligned} u(0,\tau) = u(\Lambda,\tau);\ \partial_\tau u(0,\tau) = \partial_\tau u(\Lambda,\tau) \\ (x,\tau) \in [0,\infty) \times [0,\infty). \end{aligned} \tag{4.10}$$

To reconcile these two boundary conditions, we can first set $\Lambda = L$. Now, our solution form is periodic over L as wanted.

However, the domain of x is mismatched. In the closed string case, $x \in [0, L]$ while $x \in [0,\infty]$ in the infinite sources case. We can enforce $x \in [0, L]$ to satisfy this condition. With this, the solution is:

$$\begin{aligned} u(x,\tau) = \frac{c}{2}\Theta(\tau) \sum_{n=-\infty}^{\infty} \Theta(c\tau - |x - x_0 - nL|) \\ x \in [0, L]. \end{aligned} \tag{4.11}$$

Because of the uniqueness theorem, we are guaranteed that this solution satisfies the closed string problem since it satisfies the same boundary condition.

The physical intuition for adding infinitely many sources is as follow: in the closed loop case, the string sees images of its sources propagating many times, over a period L. We can artificially reproduce this effect on the infinite string by adding infinite sources spaced by a distance L. This way, any field point will receive a signal every period L from a source nL away.

4.3 Semi-infinite String

The method of image sources can also be used to find the solution in the case of the semi-infinite string. Consider the following setup:

$$\begin{gathered}\left(\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial x^2}\right)u(x,t)=\delta(\tau)\ \delta(x-x_0)\\ u(0,\tau)=u(+\infty,\tau)=0; u=\partial_t u=0 \text{ when } \tau<0\\ x,x_0\in[0,\infty).\end{gathered} \tag{4.12}$$

To solve this, consider a solution for an infinite string with sources at $\pm x_0$:

$$u=\frac{c}{2}\Theta(\tau)(\Theta(c\tau-|x+x_0|)-\Theta(c\tau-|x-x_0|)). \tag{4.13}$$

The intuition for suggesting this solution is so that we can satisfy the condition that $u(0,\tau)=0$. Then, we realize that this solution also satisfies the other conditions, as required in the setup. By the uniqueness theorem, since this solution satisfies all conditions posed, then this must be the solution. We can also approach this problem through brute-force integration after doing a spatial-temporal Fourier Transform, but this is clearly easier once a physical intuition is achieved. Since we found the solution of the infinite string with infinite sources, it is also a natural question to find the solution for the semi-infinite string with semi-infinite sources. Superposition of solution 4.13 is the obvious approach, but we have also done a brute-force integration in Appendix B to show explicitly why it is easier to use this method of image sources (on top of showing some useful integration techniques).

5 Other boundary conditions

5.1 General Solution to the Green's Function

We have seen Green's function for the wave function under two different boundary conditions: infinite string and periodic (closed string). They were solved using different methods, but eventually connected using the image sources method. Of course, we are curious about the solution forms for different boundary conditions. If we can find the general solution the Green's function problem for the wave equation, we can then easily impose different boundary conditions.

Here, we want to find the general form of the Green's function in terms of eigenfunctions using the bilinear expansion. The bilinear expansion approach is chosen because it gives a general solution form in terms of eigenfunctions, which can be easily found. Furthermore, we only need to make sure that these eigenfunctions satisfy the boundary conditions and the Green's function will also satsify the different boundary conditions. We want to find the general solution $u(x,\tau)$ of the wave equation:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial x^2}\right) u(x,\tau) = \delta(\tau)\ \delta(x - x_0) \tag{5.1}$$

with some arbitrary boundary condition, and the causality condition: $u = 0, \partial_t u = 0$ when $\tau < 0$. Take the temporal Fourier transform $(\omega - \tau)$ to get:

$$\begin{aligned}\left(\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2} - \frac{\omega^2}{c^2}\right) \tilde{u}(x,\tau) = \delta(x - x_0) \\ \tilde{u}(x,\omega) = \int \frac{\mathrm{d}\omega}{2\pi} e^{i\omega\tau} u(x,\tau).\end{aligned} \tag{5.2}$$

Define the operator $L = \frac{\partial^2}{\partial x^2}$, the dispersion relation $k = \frac{\omega}{c}$, we now have:

$$(L - k^2)\tilde{u}(x, \omega) = \delta(x - x_0). \tag{5.3}$$

Assuming that the operator L has a complete set of eigenfunctions ϕ_n, we can use the bilinear expansion (Lemma 3.1):

$$\tilde{u} = \sum_n \frac{\phi_n(x)\phi_n^*(x_0)}{\lambda_n - \lambda}. \tag{5.4}$$

with eigenfunctions ϕ_n and eigenvalues λ_n the solutions of the eigenvalue problem (the homogeneous equation):

$$(L - \lambda_n)\, \phi_n(x) = 0. \tag{5.5}$$

Comparing the homogeneous equation (equation 5.5) to the particular case (equation 5.3), we can immediately identify $\lambda_n = k_n^2$. With this:

$$\tilde{u} = \sum_n \frac{\phi_n(x)\phi_n^*(x_0)}{k_n^2 - \frac{\omega^2}{c^2}} = -c^2 \sum_n \frac{\phi_n(x)\phi_n^*(x_0)}{\omega^2 - k^2}. \tag{5.6}$$

Performing the inverse Fourier transform on the $\omega - \tau$ pair:

$$u = -c^2 \sum_n \phi_n(x)\phi_n^*(x) \int \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega\tau} \frac{1}{\omega^2 - c^2 k_n^2}. \tag{5.7}$$

The ω-integral has been done in section 2.2. Using this result, we have:

$$u(x, \tau) = c\Theta(\tau) \sum_n \phi_n(x)\phi_n^*(x_0) \frac{\sin(c\tau k_n)}{k_n}. \tag{5.8}$$

For continuous eigenvalues (equation 4.8), this transforms to:

$$u(x,\tau) = c\Theta(\tau) \int \mathrm{d}k_n \frac{\sin(c\tau k_n)}{k_n} \sum_n \phi_n(x)^{(\alpha)} \phi_n^*(x_0)^{(\alpha)}. \tag{5.9}$$

with α degenerate modes. A discussion of degenerate modes and eigenvalues are done in section 4. With these general solution forms, all we have to do now to find specific solutions for a given boundary condition is finding eigenfunctions ϕ_n. Note that it is important to check that ϕ_n obeys the completeness relation to satisfy the bilinear expansion. The problem of finding ϕ_n is a much easier problem since equation 5.5 is a second order ordinary differential equation. Therefore, we already know the general forms of ϕ_n:

$$\phi_n(x) = A_n e^{ik_n x} + B_n e^{-ik_n x} = C_n \cos(k_n x) + D_n \sin(k_n x);\ n = 0, 1, 2, ... \tag{5.10}$$

At this point, we have the general solution of the Green's function to the wave equation. Knowing these general solution forms, let us consider some important boundary conditions:

1. Periodic boundary condition: $u(0,\tau) = u(L,\tau),\ \partial_x u(0,\tau) = \partial_x u(L,\tau)$ over some length L.
2. Dirichlet boundary condition: $u(0,\tau) = u(L,\tau) = 0$.
3. Neumann boundary condition: $\partial_x u(0,\tau) = \partial_x u(,\tau) = 0$.

5.2 Periodic and Dirichlet

Of course, we have already ϕ_n for the periodic boundary condition:

$$\begin{aligned} \phi_n^{\text{periodic}} &= \sqrt{\frac{1}{L}} e^{ik_n x} \\ k_n &= \frac{2n\pi}{L}, n \in \mathbb{Z}. \end{aligned} \tag{5.11}$$

For the Dirichlet boundary condition, we have:

$$\begin{aligned} \phi_n^{\text{Dirichlet}} &= \sqrt{\frac{2}{L}}\sin(k_n x) \\ k_n &= \frac{n\pi}{L},\ n = 1, 2, 3, ... \end{aligned} \tag{5.12}$$

These eigenfunctions are complete and we can construct a δ-function from them:

$$\frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x_0}{L}\right) = \delta(x - x_0). \tag{5.13}$$

Then, we can construct the Green's function for the wave equation satisfying the Dirichlet condition:

$$u^{\text{Dirichlet}} = \frac{2c}{L}\Theta(\tau)\sum_{n=1}^{\infty}\sin(k_n x)\sin(k_n x_0)\frac{\sin(c\tau k_n)}{k_n},\ k_n = \frac{n\pi}{L}. \tag{5.14}$$

Due to the completeness relation, this solution satisfies equation 5.3 and thus, is the Green's function to the wave equation under Dirichlet boundary condition. Furthermore, note the following relationship:

$$\phi^{\text{Dirichlet}} = \text{Im}(\phi^{\text{periodic}}) \tag{5.15}$$

where Im is the imaginary part. Here, we don't have a direct connection between solutions, but we have a connection between the eigenfunctions themselves. From there, the solution is easily formed by substitution into the general solution in terms of eigenfunctions.

5.3 The Neumann Problem

We can continue and solve the Neumann condition. Under this condition, the eigenfunctions are:

$$\begin{aligned} \phi_n^{(\text{Neumann})} &= \sqrt{\frac{2}{L}}\cos(k_n x) \\ k_n &= \frac{n\pi}{L},\ n = 0, 1, 2, 3, ... \end{aligned} \tag{5.16}$$

However, this is problematic and demonstrates why it is important to check completeness of eigenfunctions. Checking for completeness, we observe that:

$$\begin{aligned} \frac{2}{L}\sum_{n=0}^{\infty}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x_0}{L}\right) &= \frac{2}{L}\left(1+\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x_0}{L}\right)\right) \\ &= \frac{2}{L}(1+\delta(x-x_0)) \neq \delta(x-x_0). \end{aligned} \tag{5.17}$$

Note that there is an extra additive term. There is a more detailed examination of completeness using Fourier series by [15], which is reproduced here in Appendix A. Thus, the eigenfunctions we found did not satisfy the completeness relation for the Neumann condition, even though it satisfied the Neumann condition itself. We can continue to see why. Construct the bilinear expansion:

$$\tilde{u} = -\frac{2c^2}{L}\left(1+\sum_{n=1}^{\infty}\cos(k_n x)\cos(k_n x_0)\right)\frac{1}{\omega^2 - c^2 k_n^2}. \tag{5.18}$$

Obviously, $\tilde{u}$ satisfies the Neumann condition $\partial_x \tilde{u}(0,\tau) = \partial_x \tilde{u}(L,\tau) = 0$ since the extra additive vanishes under the first derivative. Now, using $\tilde{u}$ on the right hand side 5.3, we get:

$$(L - k^2)\tilde{u} = \delta(x - x_0) - \frac{1}{L}. \tag{5.19}$$

We can see here that the eigenfunctions satisfying the Neumann condition is the solution to a different differential equation than we started out. The left hand side has an extra $\frac{1}{L}$ term.

This is the same problem as we encountered in section 3.1.1 where the differential equation itself requires a modification of the source term on the right hand side. Adding a constant term on the right hand side is allowed because the Neumann condition is unique up to a constant (because it is a first derivative boundary condition). Furthermore, the physical intuition for this additional $\frac{1}{L}$ term can be interpreted similar to the extra term we encountered in section 3.1.1: a constant equals to the sum of the sources.

In addition to an additional constant, there is also a constraint equation accompanying this additional constant. The constraint equation is given in [15] in three dimension. The approach done there is similar to what we have done in section 3.1, where we performed an integral over the one dimensional domain space we are working on. In the 3D case, Gauss' Theorem is used to integrate over the space, instead of using the Fundamental Theorem of Calculus like we have done in the 1D case. Details of the derivation is provided by [15].

5.4 Relationship Between Boundary Conditions

We have seen and solved the Green's function for the wave equation under various boundary conditions. Furthermore, we have discussed the relationship between the infinite string and periodic boundary condition in detail. With the other solutions, it is customary to discuss connecting them too (even though some are quite obvious).

As we have seen before, to solve the semi-infinite string case, we only need to add two infinite string sources at $x = \pm x_0$. The superposition of solutions of infinite string at $x = \pm x_0$ ensures the condition that $u(0, \tau) = u(\infty, \tau) = 0$ and we satis-

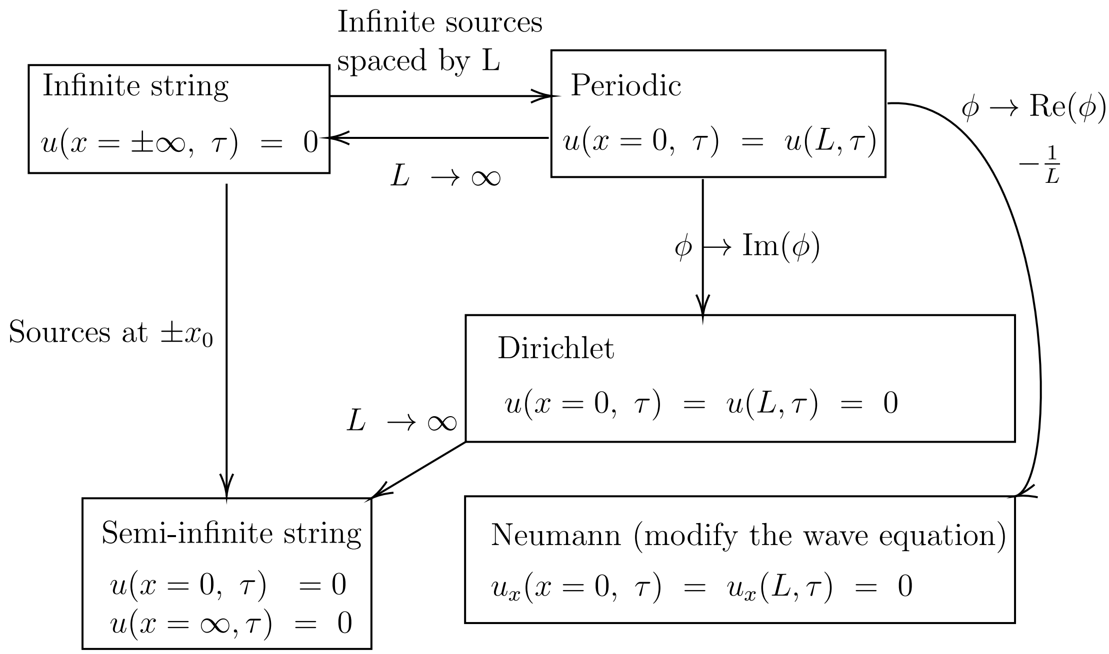

Figure 5: Relationship of different Green's function under various boundary conditions. Note that the Neumann boundary condition requires special attention since we have to modify the wave equation source term.

fied the boundary condition. This is a clear connection between the infinite string and the semi-infinite string.

In addition, we can also easily observe that: taking the imaginary part

$$\mathrm{Im}\{\phi^{\text{periodic}}\} = \sqrt{\frac{2}{L}}\sin(n\pi x/L). \tag{5.20}$$

gives us the Dirichlet eigenfunctions and the real part

$$\mathrm{Re}\{\phi^{\text{periodic}}\} = \sqrt{\frac{2}{L}}\cos(n\pi x/L). \tag{5.21}$$

gives us the Neumann eigenfunctions. Now, it is important to note that $n \in \mathbb{Z}$ for the periodic case, whereas it is $n = 0, 1, 2, 3, ...$ for the Dirichlet and Neumann cases. The reason can be reconciled by the fact that we want a set of independent eigenfunctions, but since $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$, the negative integers of n does not give us any new information. Thus, we only need them to go from $n = 0, 1, 2, 3, \ldots$.

The last method we have used to connect solutions is by taking appropriate limits. This is done when we took the limit $L \to \infty$ to go from the periodic boundary condition to the infinite string. This can also be done to connect the Dirichlet boundary condition to the semi-infinite string. This can be clearly seen since the Dirichlet boundary condition is $u(0, \tau) = u(L, \tau) = 0$ and under the limit $L \to \infty$, we recover $u(0, \tau) = u(\infty, \tau) = 0$.

6 Conclusion

The solutions for the Green's function of the wave equation under various boundary conditions were found and connected via a variety of methods. It is important to note that these were done in 1D. The original problem we had with cosmic string was of a 1D string in a 3+1D spacetime. That is, an immediate future prospect is performing this analysis in higher dimensions for a 1D string. Since we still have a 1D string and all theorems used (such as uniqueness) are still valid in higher dimensions, we speculate that most of the analysis stay relevant.

Then, we extend this analysis to the Einstein's equation. As mentioned above, the linearized Einstein's equation has another additive term involving the Riemann tensor. In addition, the linearized Einstein's equation requires a gauge choice (here, the Lorenz gauge) before taking on the final form for analysis. Since the form of the governing equation is different, it is difficult to predict if the linearized Einstein's equation suffers some of the boundary problems we have observed in the wave equation. Therefore, a careful analysis of various boundary conditions must also be done.

Finally, after establishing connections of Green's functions for the linearized Einstein equation in higher dimension for a string, we can use the known result for the closed loop string established in [2]. With that, we can find the Green's function for a cosmic string bounded between two walls, and from there, the metric perturbations for the backreaction.

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A Neumann Eigenfunctions Completeness

This derivation follows that of [15]. First, recall that we have the Neumann boundary condition and their eigenfunctions are:

$$\begin{aligned} \phi_n^{(\text{Neumann})} &= \sqrt{\frac{2}{L}}\cos(k_n x) \\ k_n &= \frac{n\pi}{L},\ n = 0, 1, 2, 3, ... \end{aligned} \tag{A.1}$$

Next, note that since ϕ_n's here are eigenfunctions, we can construct any function $f(x)$ out of them:

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n = a_0 + \sum_{n=1}^{\infty} a_n \phi_n \tag{A.2}$$

for some constant a_n. Recognizing that since ϕ_n is constructed out of cosine, this equation is in fact a Fourier series and we immediately have:

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L \mathrm{d}x\ f(x) \\ a_n &= \int_0^L \mathrm{d}x\ \phi_n(x)\phi_n^*(x_0),\ \text{for n} \neq 0 \end{aligned} \tag{A.3}$$

Therefore, the arbitrary function $f(x)$ can also be expressed as:

$$f(x) = \int_0^L \mathrm{d}x \left(\frac{1}{L} + \sum_{n=1}^{\infty} \phi_n(x)\phi_n^*(x_0)\right) f(x) \tag{A.4}$$

We can check for completeness by letting $f(x) = \delta(x)$ and we have:

$$\delta(x - x_0) \neq \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x_0}{L}\right) \tag{A.5}$$

We immediately observe that there is an extra term on the right hand side and there is a problem with the Neumann boundary condition. The rest of the discussion in section 5.3 follows.

B Semi-infinitely many sources

Consider the wave equation like equation 2.1, but now we have semi-infinitely many sources at distances $r_0, 2r_0, ...$ away. The problem is now:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u(\mathbf{r}, t) = \delta(t - t_0) \sum_{n=1}^{\infty} \delta^{(D)}(\mathbf{r} - n\mathbf{r}_0) \tag{B.1}$$

Again, let $\tau = t - t_0$.

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} - \nabla^2\right) u(\mathbf{r}, \tau) = \delta(\tau) \sum_{n=1}^{\infty} \delta^{(D)}(\mathbf{r} - n\mathbf{r}_0) \tag{B.2}$$

We similarly have boundary and causality conditions like before. And again, we can Fourier transform to approach the problem:

$$\begin{aligned} \tilde{u}(\mathbf{k}, \omega) &= \int \mathrm{d}^{(D)}\mathbf{r} \int \mathrm{d}\tau e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega\tau)}\ u(\mathbf{r}, \tau) \\ u(\mathbf{r}, \tau) &= \int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D} \int \frac{\mathrm{d}\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega\tau)}\ \tilde{u}(\mathbf{k}, \omega) \end{aligned} \tag{B.3}$$

Now for the δ functions:

$$\begin{aligned} \delta(\tau)\sum_{n=1}^{\infty} \delta^{(D)}(\mathbf{r} - n\mathbf{r}_0) &= \sum_{n=1}^{\infty} \int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D} \int \frac{\mathrm{d}\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega\tau)} e^{-in\mathbf{k}\cdot\mathbf{r}_0} \\ &= \int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D} \int \frac{\mathrm{d}\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega\tau)} \sum_{n=1}^{\infty} e^{-in\mathbf{k}\cdot\mathbf{r}_0} \\ &= \int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D} \int \frac{\mathrm{d}\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega\tau)} \frac{1}{e^{i\mathbf{k}\cdot\mathbf{r}_0} - 1} \end{aligned} \tag{B.4}$$

where in the last step, the geometric series sum was used. Using these in equation B.2, our wave equation becomes:

$$\int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D} \int \frac{\mathrm{d}\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega\tau)} (\omega^2 - c^2k^2)\tilde{u} = -c^2 \int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D} \int \frac{\mathrm{d}\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega\tau)} \frac{1}{e^{i\mathbf{k}\cdot\mathbf{r}_0} - 1}$$

$$\int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D} \int \frac{\mathrm{d}\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{R}-\omega\tau)} \left[(\omega^2 - c^2k^2)\tilde{u} + \frac{c^2}{e^{i\mathbf{k}\cdot\mathbf{r}_0} - 1}\right] = 0$$

$$\implies \tilde{u}(\mathbf{k},\omega) = -\frac{c^2}{\omega^2 - c^2k^2} \cdot \frac{1}{e^{i\mathbf{k}\cdot\mathbf{r}_0} - 1} \tag{B.5}$$

Again, we have two poles at $\omega = \pm ck$. The extra factor did not have any effects on the ω complex plane. Thus, the ω-integral result from the single source problem still applies and we now have:

$$u(\mathbf{r},\tau) = c\Theta(\tau) \int \frac{\mathrm{d}^{(D)}\mathbf{k}}{(2\pi)^D}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, \frac{\sin(c\tau k)}{k} \cdot \frac{1}{e^{i\mathbf{k}\cdot\mathbf{r}_0} - 1} \tag{B.6}$$

B.1 The k-integral in 1D

In one dimension, this integral becomes:

$$u(r,\tau) = c\Theta(\tau) \int \frac{\mathrm{d}k}{2\pi}\, e^{ikr}\, \frac{\sin(c\tau k)}{k} \cdot \frac{1}{e^{ikr_0} - 1} := c\Theta(\tau) I \tag{B.7}$$

At this point, two directions can (potentially) be taken to evaluate this integral I. The first is using the Fourier convolution theorem $\mathcal{F}^{-1}(f \cdot g) = \mathcal{F}^{-1}(f) * \mathcal{F}^{-1}(g)$. Identifying $f = \frac{\sin(c\tau k)}{k}$, then $\mathcal{F}^{-1}(f) = \frac{1}{2}\Theta(c\tau - |r|)$. However, the difficulty arises when performing $\mathcal{F}^{-1}(g)$ for $g = \frac{1}{e^{ikr_0}-1}$. And yet after that, we still have another (potentially tricky) integral to evaluate since $*$ is a convolution.

The second method is brute forcing through with complex analysis, which is what will be attempted. The next 5 pages will be just performing this integration. The solution is at equation B.22.

First, perform some simplifications:

$$\frac{e^{ikr}}{e^{ikr_0}-1} = \frac{e^{ikr}}{\exp\left(\frac{ikr_0}{2}\right)\left[\exp\left(\frac{ikr_0}{2}\right) - \exp\left(-\frac{ikr_0}{2}\right)\right]} = \frac{e^{ik(r-r_0/2)}}{2i\sin(kr_0/2)}$$
$$= -i\frac{\cos(k(r-r_0/2))}{2\sin(kr_0/2)} + \frac{\sin(k(r-r_0/2))}{2\sin(kr_0/2)}$$

$$\implies I = -i\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{\cos(k(r-r_0/2))\sin(c\tau k)}{2\sin(kr_0/2)} + \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{\sin(k(r-r_0/2))\sin(c\tau k)}{2\sin(kr_0/2)}$$

The first integrand is an odd function, thus vanishing on the integral across $(-\infty,\infty)$. Now with the second integral, we can use the trigonometric identity for $\sin(a-b)$ to expand $\sin(k(r-r_0/2))$:

$$\begin{aligned} I &= \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\left(\frac{\sin(kr)\cos\left(\frac{kr_0}{2}\right)\sin(c\tau k)}{2\sin\left(\frac{kr_0}{2}\right)} - \frac{\cos(kr)\sin\left(\frac{kr_0}{2}\right)\sin(c\tau k)}{2\sin\left(\frac{kr_0}{2}\right)}\right) \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\sin(c\tau k)\sin(kr)\cot\left(\frac{kr_0}{2}\right)\frac{1}{k} - \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\cos(kr)\sin(c\tau k)\frac{1}{k} \\ &:= \frac{1}{2}I_1 - \frac{1}{2}I_2 \end{aligned} \tag{B.8}$$

We have actually seen integral I_2 before. It is the exact same form as the k-integral in section 2.3. Therefore:

$$I_2 = \frac{1}{2}\Theta(c\tau - |r|) \tag{B.9}$$

Observing I_1's integrand, we see periodic, isolated singularities due to cot. To solve I_1, the first step is expand cot using:

$$\cot(ax) = ax \sum_{n=-\infty}^{\infty} \frac{1}{(ax)^2 - \pi^2 n^2}$$

$$I_1 = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \sin(c\tau k)\sin(kr)) \frac{1}{k}\frac{kr_0}{2} \sum_{n=-\infty}^{\infty} \frac{1}{(\frac{kr_0}{2})^2 - \pi^2 n^2} = \frac{r_0}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\sin(c\tau k)\sin(kr))}{(\frac{kr_0}{2})^2 - \pi^2 n^2}$$
$$= \frac{2}{r_0}\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \sum_{n=-\infty}^{\infty} \frac{\sin(c\tau k)\sin(kr))}{k^2 - \left(\frac{2\pi n}{2}\right)^2} = \frac{2}{r_0} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{\sin(c\tau k)\sin(kr))}{k^2 - \left(\frac{2\pi n}{2}\right)^2}$$

There are now poles at $k = \frac{2n\pi}{r_0}$ of order 1, *except* at $k = 0$. We can rewrite the integral to emphasize this fact:

$$I_1 = \frac{1}{\pi r_0}\left(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k \frac{\sin(kr)\sin(c\tau k)}{k^2 - \left(\frac{2n\pi}{r_0}\right)^2} + \int_{-\infty}^{\infty} \mathrm{d}k \ \frac{\sin(kr)\sin(c\tau k)}{k^2} \right) := \frac{1}{\pi r_0}(I_3 + I_4) \tag{B.10}$$

Let us perform I_3 first. To proceed, we use the trigonometric identity

$$\sin(kr)\sin(c\tau k) = \frac{1}{2}(\cos(k(c\tau - r)) - \cos(c\tau + r)k = \frac{1}{2}\operatorname{Re}(e^{ika_-} - e^{ika_+})$$

where we have defined:

$$a_- := c\tau - r, \ a_+ = c\tau + r \tag{B.11}$$

$$I_3 = \frac{1}{2}\operatorname{Re}\left(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k \frac{e^{ika_-}}{k^2 - \left(\frac{2n\pi}{r_0}\right)^2} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k \frac{e^{ika_+}}{k^2 - \left(\frac{2n\pi}{r_0}\right)^2} \right) := \frac{1}{2}\operatorname{Re}(I_5 - I_6) \tag{B.12}$$

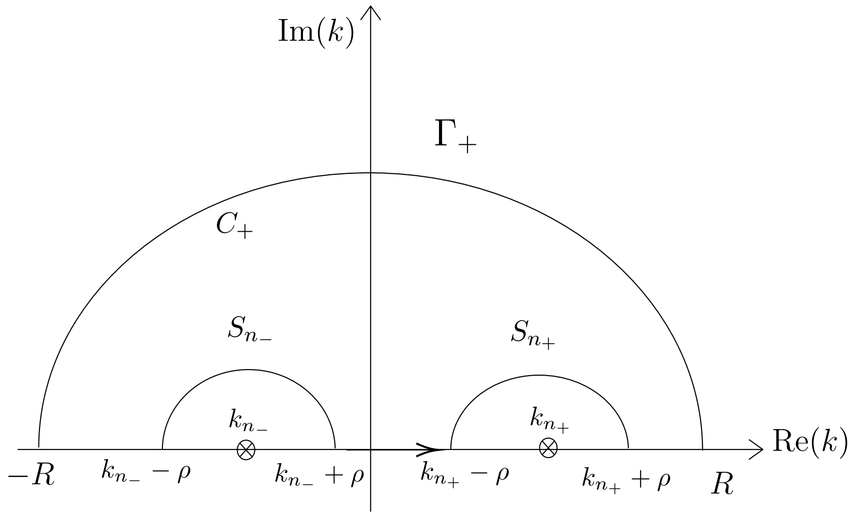

Figure 6: The idented contour Γ_+.

Let us do I_5 (don't worry, after this one we are pretty much done).

$$I_5 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k \frac{e^{ika_-}}{k^2 - \left(\frac{2n\pi}{r_0}\right)^2} = \lim_{N\longrightarrow\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \int_{-\infty}^{\infty} \mathrm{d}k \frac{e^{ika_-}}{\left(k - \frac{2n\pi}{r_0}\right)\left(k + \frac{2n\pi}{r_0}\right)}$$

Now, at *each* n, there are only two poles per integral: $k_{n_-} = -\frac{2n\pi}{r_0}$, $k_{n_+} = \frac{2n\pi}{r_0}$. Since these poles are on the real axis, we can draw an indented contour to evaluate this integral. Consider the contour Γ_+ in Figure 6. This contour consists of a big semicircle, C_+ with radius $R = N + \epsilon$ (for small ϵ so that the big semicircle covers all poles) centering at 0, two smaller semicircles with radius ρ at the two poles k_{n_-} and k_{n_+}, and straight lines connecting these semicircles. These new parameters has to be taken to their appropriate limits. Namely, $\rho \longrightarrow 0$ and $R \longrightarrow \infty$, to recover the original integral along the real line. Integrating along this contour, we get:

$$\lim_{\substack{\rho\longrightarrow 0\\ R\longrightarrow\infty}}\left(\int_{-R}^{k_{n_-}-\rho}+\int_{k_{n_-}+\rho}^{k_{n_+}-\rho}+\int_{k_{n_+}+\rho}^{R}+\int_{S_{n_-}}+\int_{S_{n_+}}+\int_{C_+}\right)dk\frac{e^{ika_-}}{\left(k-\frac{2n\pi}{r_0}\right)\left(k+\frac{2n\pi}{r_0}\right)}$$
$$=\lim_{\substack{\rho\longrightarrow 0\\ R\longrightarrow\infty}}\oint_{\Gamma_+}dk\frac{e^{ika_-}}{\left(k-\frac{2n\pi}{r_0}\right)\left(k+\frac{2n\pi}{r_0}\right)}=0 \tag{B.13}$$

Here, the right hand side vanishes due to Cauchy's Integral Theorem because the region inside the contour Γ_+ is smooth.

Now, evaluate each integrals with their appropriate limits. First, using Jordan's Lemma by assuming that $a_- > 0$ (if $a_- < 0$, we can just draw a similar indented contour but on the lower half plane, not much will change):

$$\lim_{R\longrightarrow\infty}\int_{C_+}dk\frac{e^{ika_-}}{\left(k-\frac{2n\pi}{r_0}\right)\left(k+\frac{2n\pi}{r_0}\right)}=0 \tag{B.14}$$

Lemma B.1. *[11] If f has a single pole at $z=c$ and Tr is the circular arc such that $Tr: z=c+re^{i\theta}$, $(\theta_1\leq\theta\leq\theta_2)$, then:*

$$\lim_{r\longrightarrow 0^+}\int_{Tr}f(z)dz=i(\theta_2-\theta_1)\mathrm{Res}(f;c)$$

.

Remark: *For a clockwise semicircle, S_r then $\theta_2-\theta_1=-\pi$, then:*

$$\lim_{r\longrightarrow 0+}\int_{S_r}f(z)dz=-i\pi\mathrm{Res}(f;c)$$

Now, for the two semicircles S_{n_+} and S_{n_-}, we will use Lemma B.1 (specifically the remark):

$$\lim_{\rho\longrightarrow 0}\int_{S_{n_-}} dk\frac{e^{ika_-}}{\left(k-\frac{2n\pi}{r_0}\right)\left(k+\frac{2n\pi}{r_0}\right)} = -i\pi \mathrm{Res}\left(-\frac{2n\pi}{r_0}\right) = \frac{ir_0}{4n}\exp\left(\frac{-2n\pi ika_-}{r_0}\right)$$
$$\lim_{\rho\longrightarrow 0}\int_{S_{n_+}} dk\frac{e^{ika_-}}{\left(k-\frac{2n\pi}{r_0}\right)\left(k+\frac{2n\pi}{r_0}\right)} = -i\pi \mathrm{Res}\left(\frac{2n\pi}{r_0}\right) = -\frac{ir_0}{4n}\exp\left(\frac{2n\pi ika_-}{r_0}\right) \tag{B.15}$$

And finally, the remaining pieces give us back our integral after the appropriate limits:

$$\begin{aligned}\lim_{\substack{\rho\longrightarrow 0\\ R\longrightarrow\infty}}\left(\int_{-R}^{k_{n_-}-\rho}+\int_{k_{n_-}+\rho}^{k_{n_+}-\rho}+\int_{k_{n_+}+\rho}^{R}\right)\mathrm{d}k\frac{e^{ika_-}}{\left(k-\frac{2n\pi}{r_0}\right)\left(k+\frac{2n\pi}{r_0}\right)}\\ =\int_{-\infty}^{\infty}\mathrm{d}k\frac{e^{ika_-}}{\left(k-\frac{2n\pi}{r_0}\right)\left(k+\frac{2n\pi}{r_0}\right)}\end{aligned} \tag{B.16}$$

Combining the results from B.14, B.15, and B.16 in B.13, we get:

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{d}k\frac{e^{ika_-}}{\left(k-\frac{2n\pi}{r_0}\right)\left(k+\frac{2n\pi}{r_0}\right)} &= \frac{ir_0}{4n}\left(\exp\left(\frac{2n\pi ika_-}{r_0}\right)-\exp\left(\frac{-2n\pi ika_-}{r_0}\right)\right)\\ &= -\frac{r_0}{2n}\sin\left(\frac{2n\pi ika_-}{r_0}\right)\end{aligned} \tag{B.17}$$

With this, I_5 becomes:

$$
\begin{aligned}
I_5 &= \lim_{N \longrightarrow \infty} \sum_{\substack{n=-N \\ n \neq 0}}^{N} \int_{-\infty}^{\infty} \mathrm{d}k \frac{e^{ika_-}}{\left(k - \frac{2n\pi}{r_0}\right)\left(k + \frac{2n\pi}{r_0}\right)} \\
&= -\frac{r_0}{2} \lim_{N \longrightarrow \infty} \sum_{\substack{n=-N \\ n \neq 0}}^{N} \frac{1}{n} \sin\left(\frac{2n\pi ika_-}{r_0}\right)
\end{aligned}
\tag{B.18}
$$

From here, we can get I_6 easily, since I_6 is identical to I_5 but with a_+ instead of a_-. Therefore:

$$
\begin{aligned}
I_6 &= \lim_{N \longrightarrow \infty} \sum_{\substack{n=-N \\ n \neq 0}}^{N} \int_{-\infty}^{\infty} \mathrm{d}k \frac{e^{ika_+}}{\left(k - \frac{2n\pi}{r_0}\right)\left(k + \frac{2n\pi}{r_0}\right)} \\
&= -\frac{r_0}{2} \lim_{N \longrightarrow \infty} \sum_{\substack{n=-N \\ n \neq 0}}^{N} \frac{1}{n} \sin\left(\frac{2n\pi ika_+}{r_0}\right)
\end{aligned}
\tag{B.19}
$$

From here, we get I_3:

$$
\begin{aligned}
I_3 &= \frac{1}{2}\,\mathrm{Re}(I_5 - I_6) \\
&= \frac{r_0}{4}\lim_{N\longrightarrow\infty}\left[\sum_{\substack{n=-N\\n\neq 0}}^{N}\frac{1}{n}\sin\!\left(\frac{2n\pi ika_+}{r_0}\right) - \sum_{\substack{n=-N\\n\neq 0}}^{N}\frac{1}{n}\sin\!\left(\frac{2n\pi ika_-}{r_0}\right)\right] \\
&= \frac{r_0}{4}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{1}{n}\left[\sin\!\left(\frac{2n\pi ik(c\tau + r)}{r_0}\right) - \sin\!\left(\frac{2n\pi ik(c\tau - r)}{r_0}\right)\right] \\
&= \frac{r_0}{2}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{1}{n}\cos\!\left(\frac{2n\pi c\tau k}{r_0}\right)\sin\!\left(\frac{2n\pi kr}{r_0}\right) \\
&= \frac{r_0}{2}\left(\sum_{n=-\infty}^{-1}\frac{1}{n}\cos\!\left(\frac{2n\pi c\tau k}{r_0}\right)\sin\!\left(\frac{2n\pi kr}{r_0}\right) + \sum_{n=1}^{\infty}\frac{1}{n}\cos\!\left(\frac{2n\pi c\tau k}{r_0}\right)\sin\!\left(\frac{2n\pi kr}{r_0}\right)\right) \\
&= \frac{r_0}{2}\left(\sum_{n=1}^{\infty}-\frac{1}{n}\cos\!\left(-\frac{2n\pi c\tau k}{r_0}\right)\sin\!\left(-\frac{2n\pi kr}{r_0}\right) + \sum_{n=1}^{\infty}\frac{1}{n}\cos\!\left(\frac{2n\pi c\tau k}{r_0}\right)\sin\!\left(\frac{2n\pi kr}{r_0}\right)\right) \\
&= r_0\sum_{n=1}^{\infty}\frac{1}{n}\cos\!\left(\frac{2n\pi c\tau k}{r_0}\right)\sin\!\left(\frac{2n\pi kr}{r_0}\right)
\end{aligned}
$$

We have one last integral to perform, I_4. Thankfully, this is a known result:

$$
I_4 = \int_{-\infty}^{\infty}\mathrm{d}k\,\frac{\sin(kr)\sin(c\tau k)}{k^2} = \frac{\pi}{2}(|a_+| - |a_-|) \tag{B.20}
$$

with a_+ and a_- defined in equation B.11. With this, we have I_1:

$$
I_1 = \frac{1}{\pi r_0}(I_3 + I_4) = \frac{|a_+| - |a_-|}{2r_0} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\!\left(\frac{2n\pi c\tau k}{r_0}\right)\sin\!\left(\frac{2n\pi kr}{r_0}\right) \tag{B.21}
$$

Since $I = \frac{1}{2}(I_1 - I_2)$, we now have the full solution to the integral I. The full solution for the 1D case with semi-infinitely many sources:

$$u_2(r,\tau) = \frac{c}{2}\Theta(\tau)\left[\sum_{n=1}^{\infty}\frac{1}{n\pi}\cos\left(\frac{2n\pi c\tau k}{r_0}\right)\sin\left(\frac{2n\pi kr}{r_0}\right) + \frac{|c\tau + r| - |c\tau - r|}{2r_0} - \frac{\Theta(c\tau - |r|)}{2}\right] \tag{B.22}$$

Note that the last term is in fact the solution u_1 from the single source case. We can also notice that the term in the sum is potentially derived from an eigenfunction expansion.