AST 1420

Galactic Structure and Dynamics

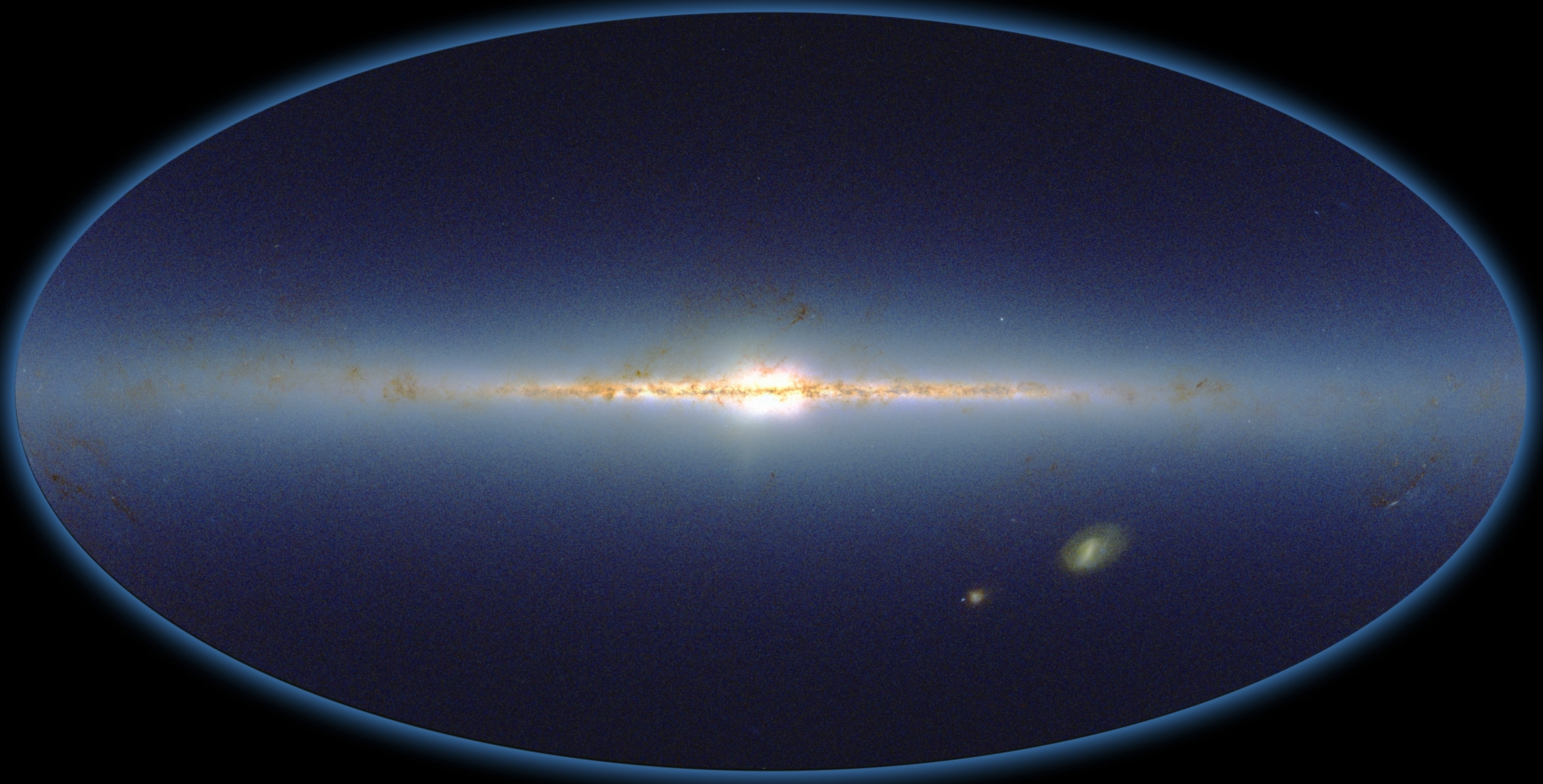
Presentations

- Week 10: Mar 18, date/time TBD
- Each student presents on a topic for ~20 min.
- Encouraged to find your own topic in Galactic structure and dynamics!
    - Should be tied to a recent paper (2022-2024)
    - But explore topic more broadly than just this paper
- **Please email me with your proposed topic by Feb. 14**

NGC 5907

M31

2MASS Covers the Sky



The Two Micron All Sky Survey

Infrared Processing and Analysis Center/Caltech & Univ. of Massachusetts

Computing the gravitational potential for a disk

- Simple: just solve the Poisson equation!

$$\nabla^2 \Phi = 4\pi G \rho.$$

- Newton's theorems don't apply

Newton's first theorem: *A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.*

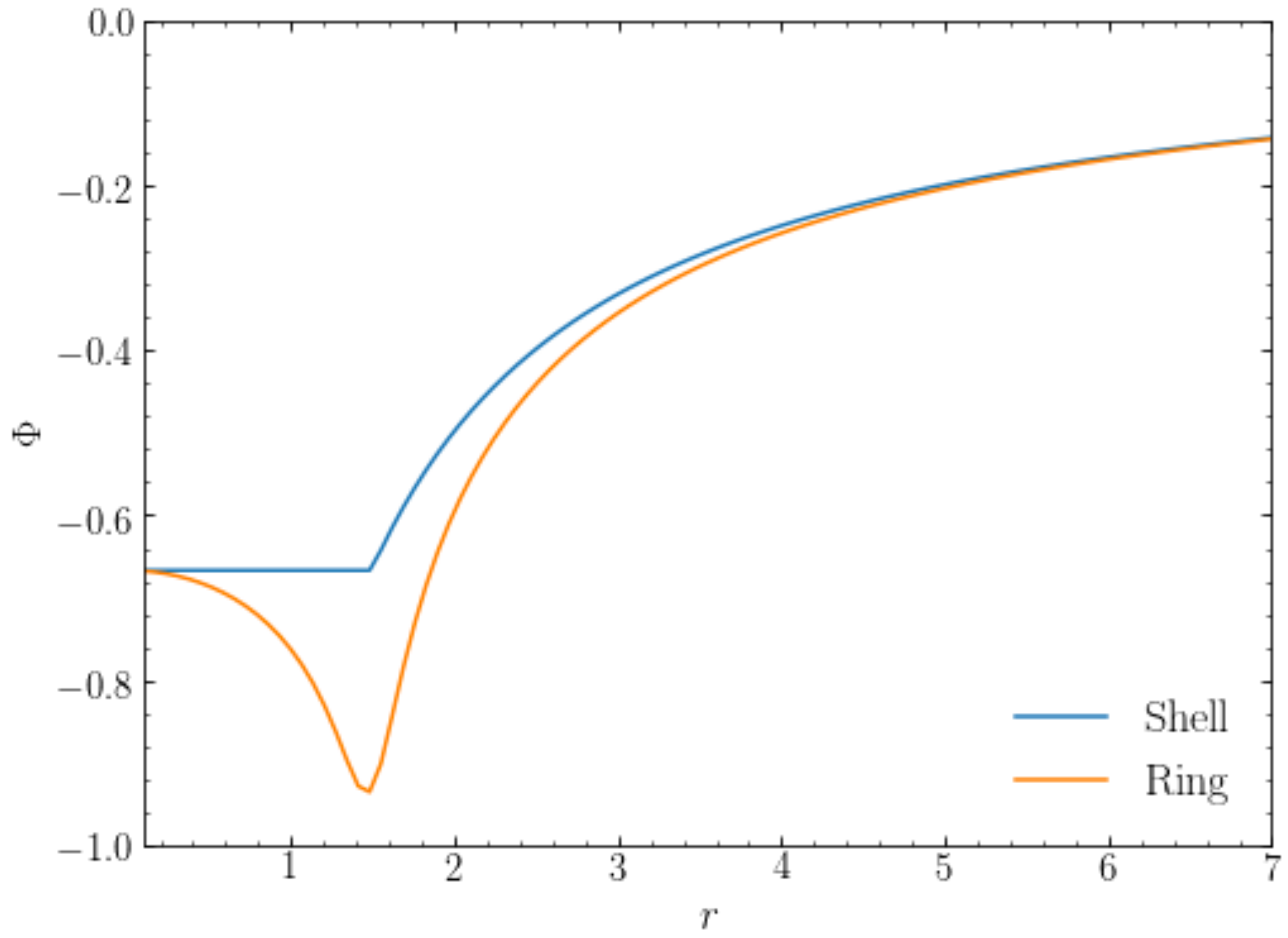
Newton's second theorem: *The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all of the shell's matter were concentrated into a point at its center.*

Computing the gravitational potential for a disk

- Simple: just solve the Poisson equation!

$$\nabla^2\Phi = 4\pi G\rho.$$

- Newton's theorems don't apply

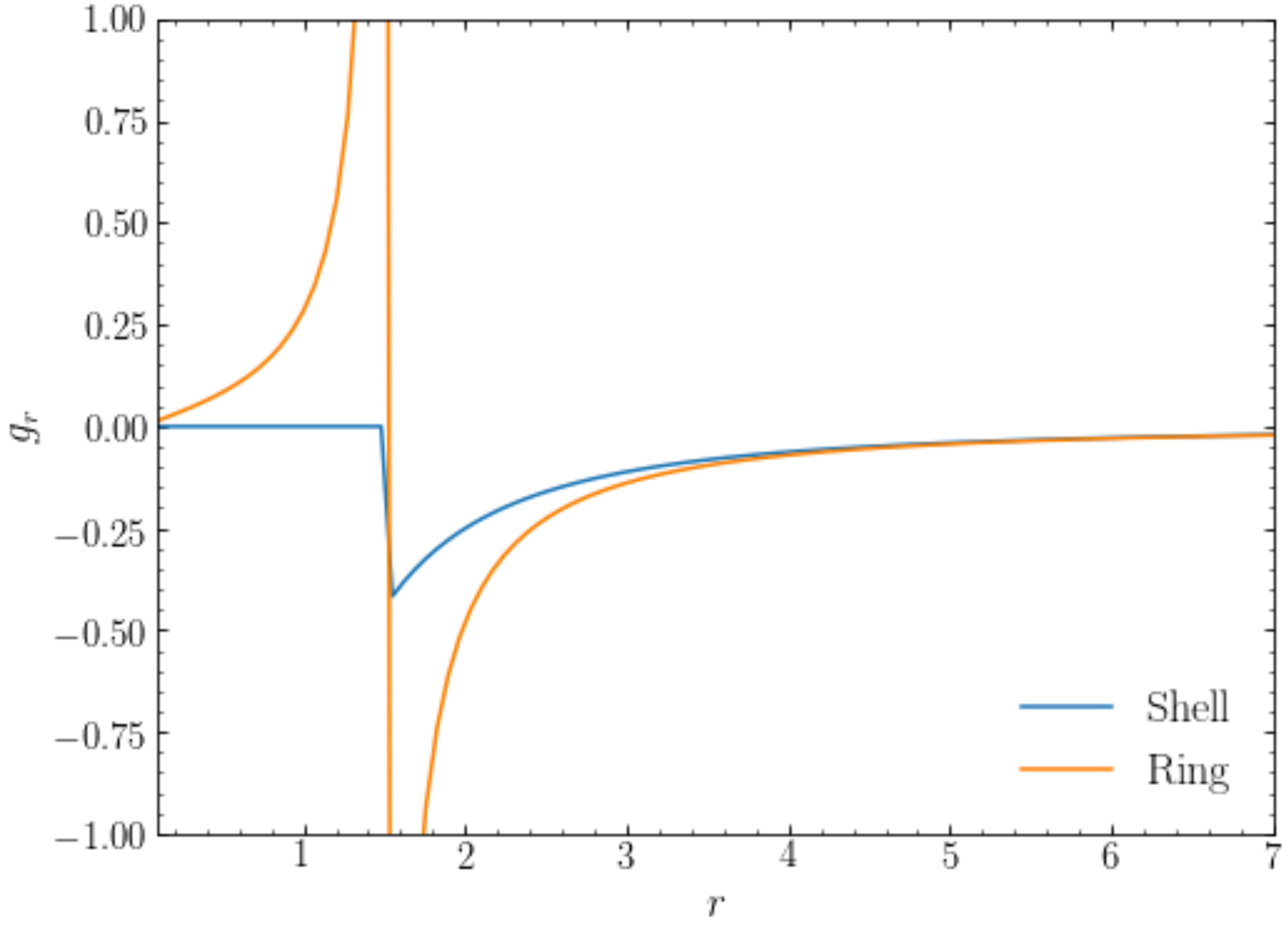


Computing the gravitational potential for a disk

- Simple: just solve the Poisson equation!

$$\nabla^2\Phi = 4\pi G\rho.$$

- Newton's theorems don't apply



Razor-thin disks

- ‘Extreme’ assumption: disk is infinitely thin

$$\rho(R, \phi, z) = \Sigma(R, \phi)\, \delta(z)\,,$$

- Integrate Poisson equation

$$\int_{\mathcal{V}} \mathrm{d}\mathbf{x}\, \nabla^2 \Phi(\mathbf{x}) = 4\pi G \int_{\mathcal{V}} \mathrm{d}\mathbf{x}\, \rho(\mathbf{x})\,,$$

- Divergence theorem

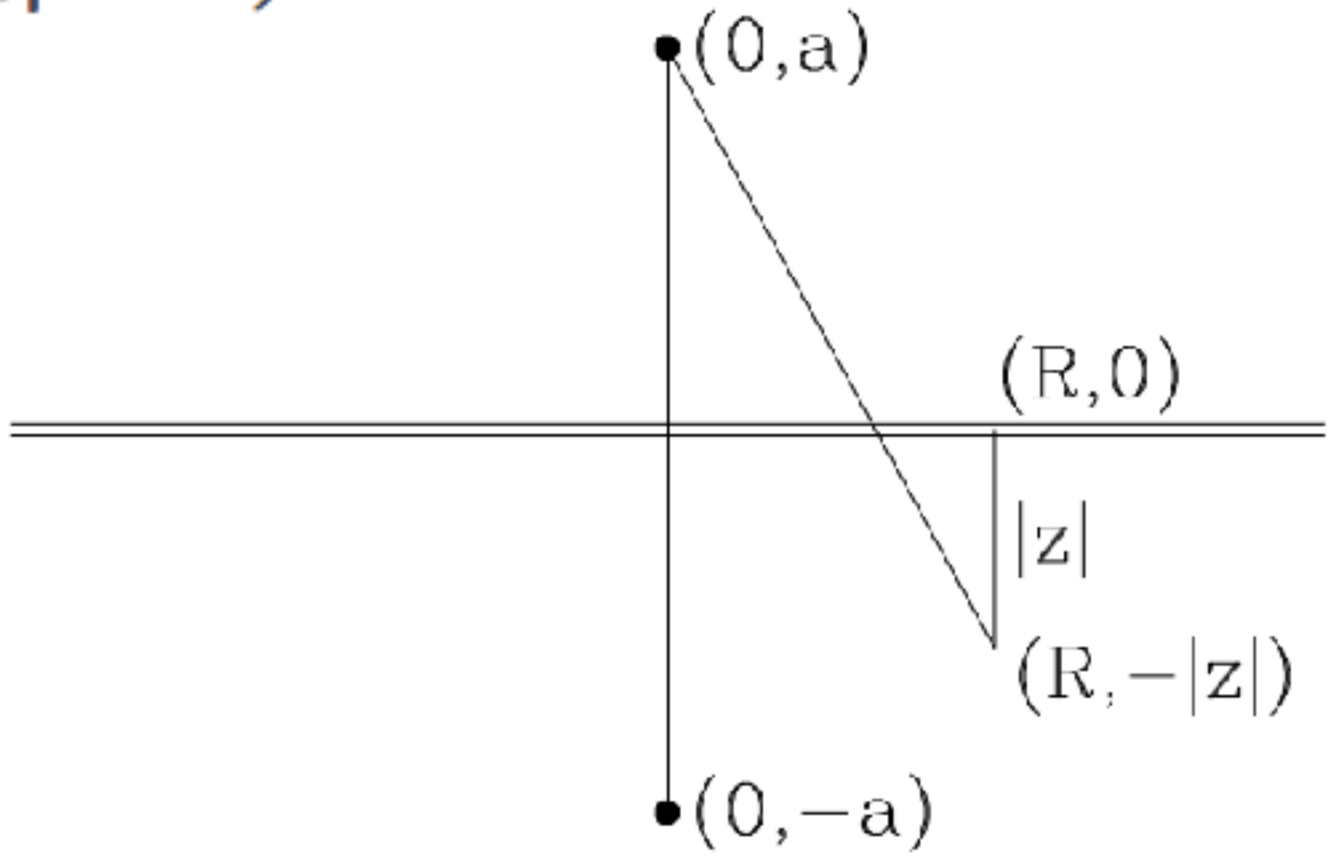
$$\int_S \mathrm{d}S \cdot \nabla\Phi(R, z) = 4\pi G \int R\, \mathrm{d}R\, \mathrm{d}\phi\, \Sigma(R)\,.$$

- For small volume

$$\pm \left.\frac{\partial \Phi(R, z)}{\partial z}\right|_{z=0\pm} = 2\pi G\, \Sigma(R)\,,$$

Example: Kuzmin disk

$$\Phi(R, z) = -\frac{GM}{\sqrt{R^2 + (|z| + a)^2}} ,$$



- At any z > 0: point mass at (0,-a)
- At any z < 0: point mass at (0,a)
- For any z =/= 0 —> density = 0 —> razor-thin
- At z=0 ~ Plummer —> non-zero density

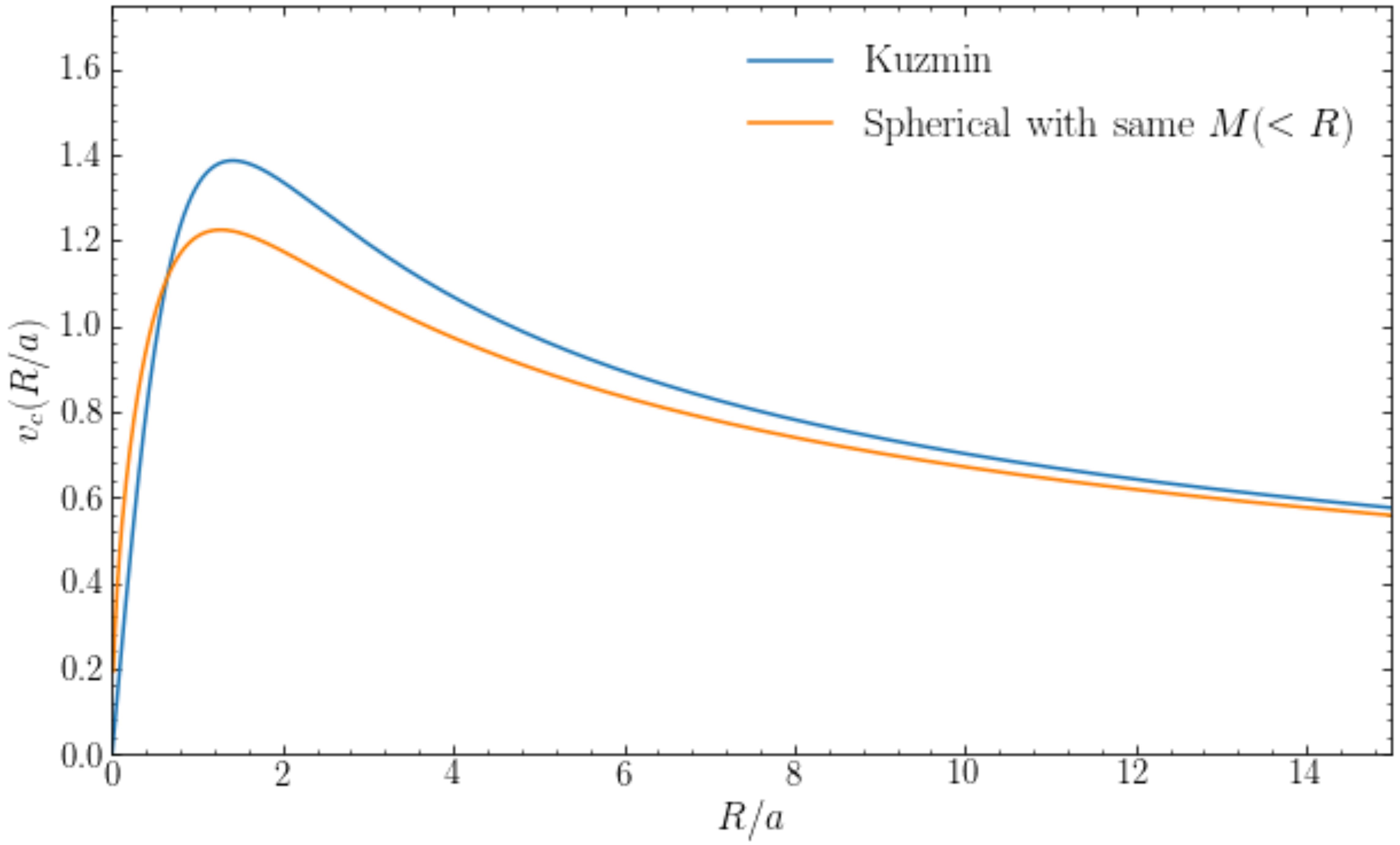
$$\Sigma(R) = \frac{M\,a}{2\pi} \frac{1}{\left(R^2 + a^2\right)^{3/2}} .$$

Example: Kuzmin disk

$$v_c(R) = \sqrt{\frac{G\,M\,R^2}{\left(R^2 + a^2\right)^{3/2}}}\,.$$

$$M(< R) = \frac{M\,R}{\sqrt{R^2 + a^2}}$$

$$R_{\rm peak} = \sqrt{2}\,a\,.$$



Thick disks: Miyamoto-Nagai

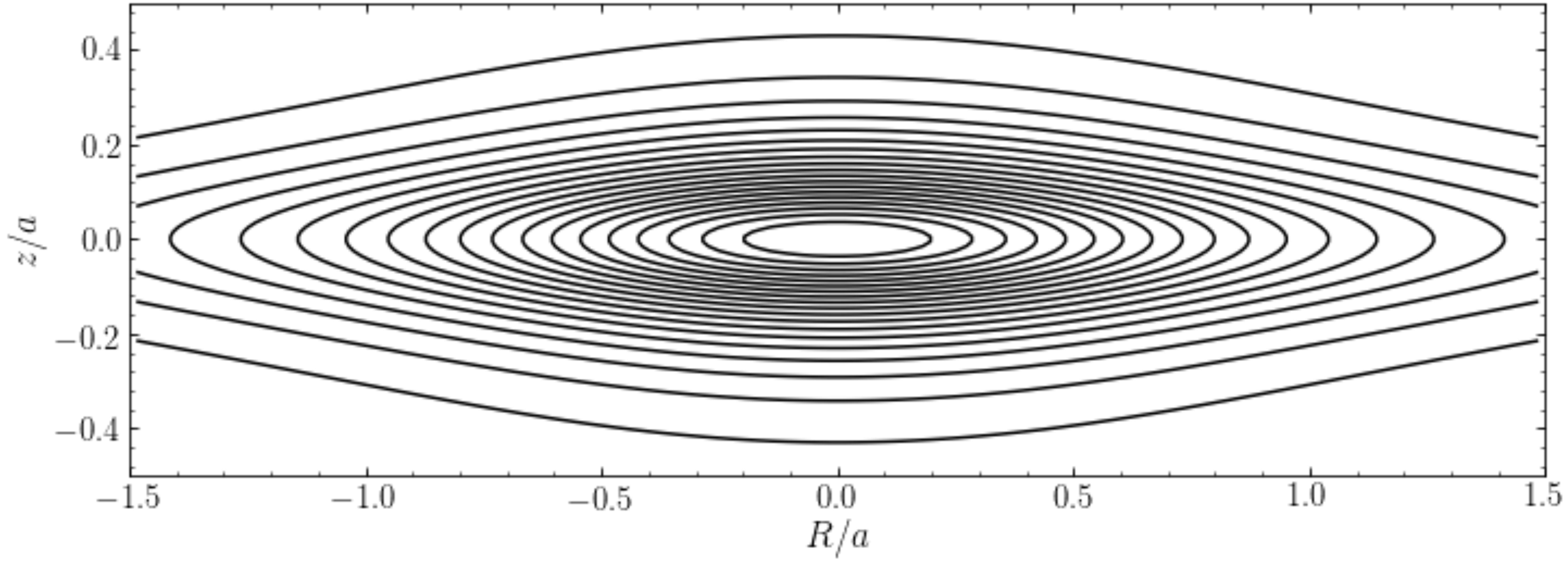
- Razor-thin is not realistic for orbits in disks
- Can thicken Kuzmin by replacing |z| —> √[z^2 + b^2]
- Miyamoto-Nagai:

$$\Phi(R, z) = -\frac{GM}{\sqrt{R^2 + (\sqrt{z^2 + b^2} + a)^2}}$$

- b —> 0: Kuzmin
- a —> 0: Plummer (spherical)

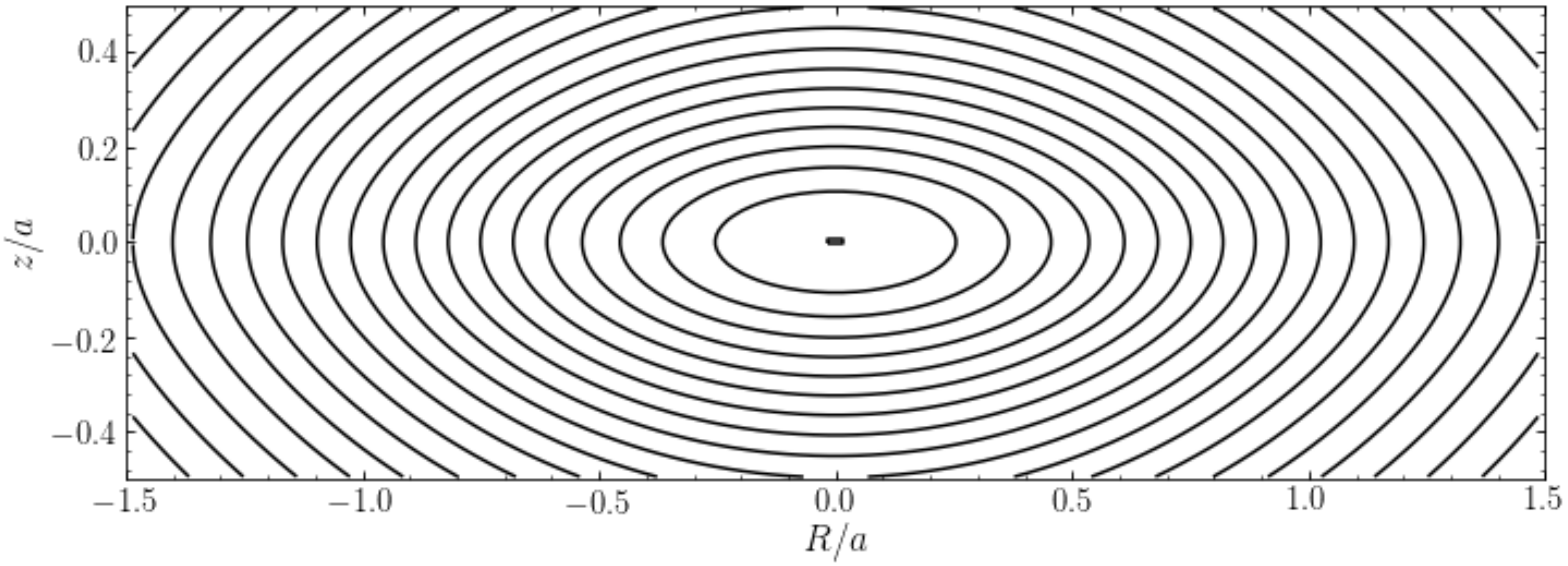
Thick disks: Miyamoto-Nagai

$$\rho(R,z) = \left(\frac{b^2 M}{4\pi}\right) \frac{a R^2 + \left(3\sqrt{z^2+b^2}+a\right)\left(\sqrt{z^2+b^2}+a\right)^2}{\left(R^2+(\sqrt{z^2+b^2}+a)^2\right)^{5/2}\left(z^2+b^2\right)^{3/2}} .$$



Thick disks: Miyamoto-Nagai

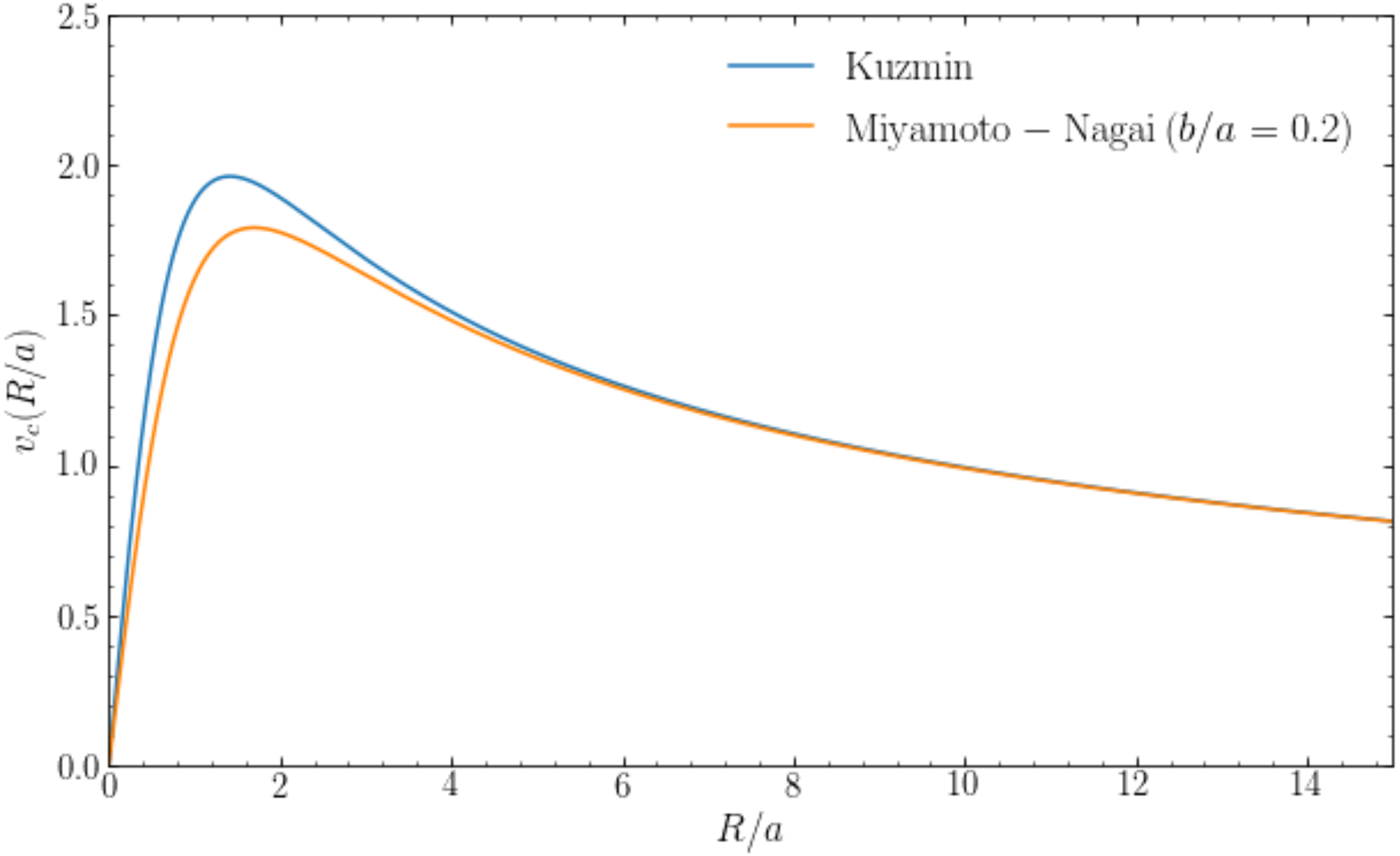
Potential is much less flattened:



Thick disks: Miyamoto-Nagai

- Advantages:
    - Analytical formula —> fast for, e.g., orbit integration
    - Can vary b/a to get different ‘thicknesses’
- Disadvantages:
    - Large R ~ R^{-3} —> not like a realistic disk, too much density at large R
    - Similarly, vertical profile not realistic

Miyamoto-Nagai rotation curve



Flattened + flat rotation curve

- Spherical model:

$$\Phi(r) = v_c^2 \ \ln r$$

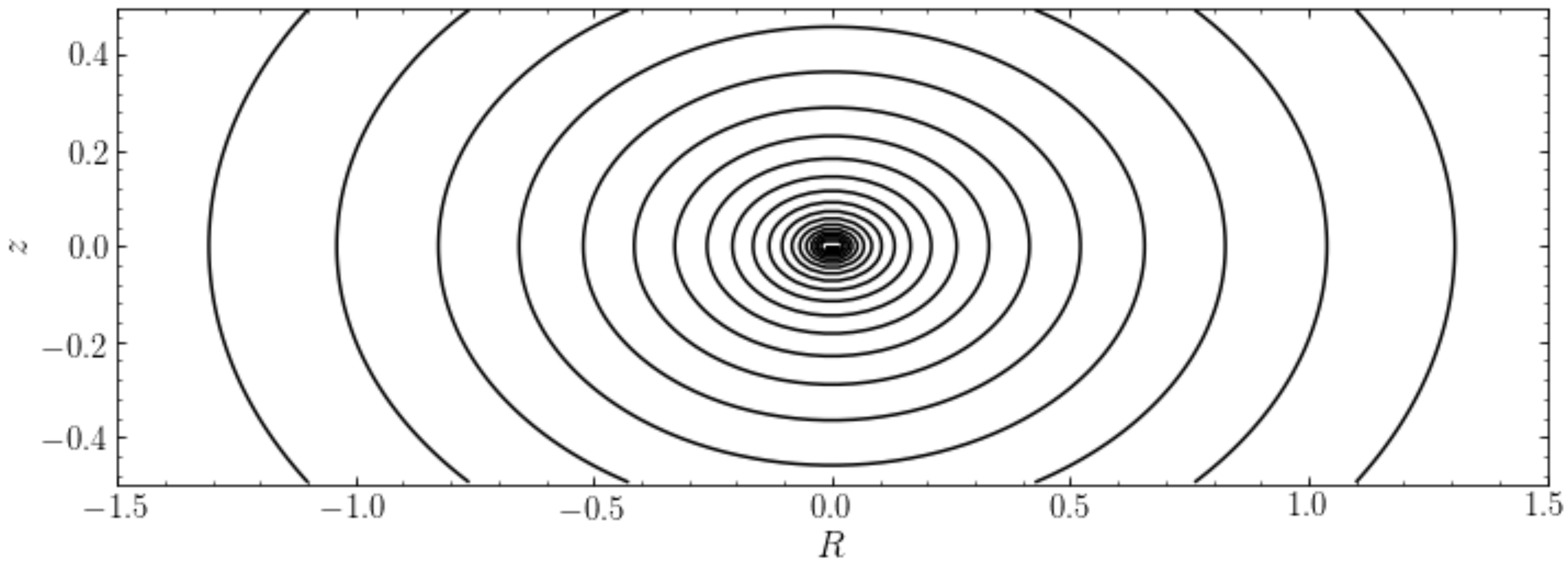
- Has $v_c(r)$ = constant
- General flattening strategy: r —> m = $\sqrt{[R^2 + z^2/q^2]}$

$$\Phi(R, z) = \frac{v_c^2}{2} \ln\left(R^2 + \frac{z^2}{q^2} \right)$$

- Has $v_c(R)$ = constant at z=0

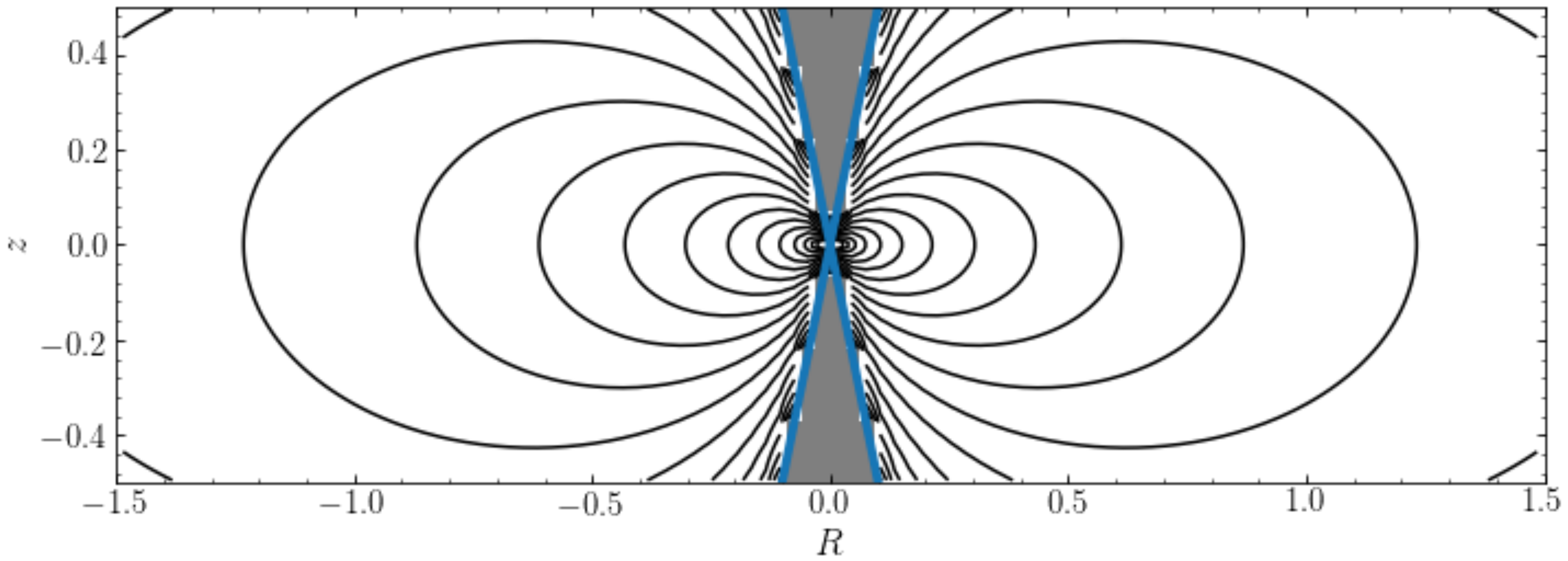
Flattened logarithmic potential

- Potential for q=0.7 is slightly flattened:



Flattened logarithmic potential

- But density becomes negative around R=0!



Flattened logarithmic potential: density

$$\rho(R,z) = \frac{v_c^2}{4\pi\, G\, q^2} \frac{R^2 + 2\,z^2 - z^2/q^2}{\left(R^2 + z^2/q^2\right)^2}$$

- When q < 1/√2: density is negative at R=0
- and everywhere between

$$z = \pm R/\sqrt{1/q^2 - 2}.$$

- Thus, you cannot flatten a logarithmic potential too much without creating negative densities
- General problem when flattening a potential using r —> m = √[R^2 + z^2/q^2] [e.g., often used to get flattened DM halos]

Potentials for disk densities

Double-exponential disk model

- First week:
  - radial surface brightness ~ exp(-R/R_d)
  - vertical surface brightness ~ $\mathrm{sech}^2[z/2z_d]$ ~ exp(-|z|/z_d)
- Good model for galactic disk is therefore

$$\rho(R, z) = \Sigma(0)\, e^{-R/R_d}\, \frac{1}{2\, z_d}\, e^{-|z|/z_d} = \Sigma(0)\, e^{-\alpha R}\, \frac{\beta}{2}\, e^{-\beta|z|}$$

Poisson equation for *axisymmetric razor-thin* disks

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\,\frac{\partial\Phi(R,z)}{\partial R}\right)+\frac{\partial^2\Phi(R,z)}{\partial z^2}=4\pi\,G\,\Sigma(R)\,\delta(z)$$

- Solution:

$$\tilde{\Phi}(R;k)=-2\pi\,G\,e^{-k|z|}\,J_0(kR)$$

- With Σ

$$\tilde{\Sigma}(R;k)=k\,J_0(kR)\,.$$

- Potential-density pair —> complete?

Poisson equation for *axisymmetric razor-thin* disks

- If we can decompose

$$\Sigma(R) = \int_0^\infty \mathrm{d}k\, J_0(kR)\, k\, S_0(k)\,,$$

- Then the potential is

$$\begin{aligned}\Phi(R,z) &= -2\pi\, G \int_0^\infty \mathrm{d}k\, e^{-k|z|}\, J_0(kR)\, S_0(k) \\ &= -2\pi\, G \int_0^\infty \mathrm{d}k\, e^{-k|z|}\, J_0(kR) \int_0^\infty \mathrm{d}R'\, J_0(kR')\, R'\, \Sigma(R')\,.\end{aligned}$$

- Decomposition from the Fourier-Bessel theorem

$$S_0(k) = \int_0^\infty \mathrm{d}R'\, J_0(kR')\, R'\, \Sigma(R')$$

Circular velocity for *axisymmetric razor-thin* disks

$$\Phi(R,0) = -2\pi G \int_0^\infty \mathrm{d}k\, J_0(kR)\, S_0(k)$$
$$= -2\pi G \int_0^\infty \mathrm{d}k\; J_0(kR) \int_0^\infty \mathrm{d}R'\, J_0(kR')\, R'\, \Sigma(R')\,.$$

$$v_c^2(R) = 2\pi G R \int_0^\infty \mathrm{d}k\, J_1(kR)\, k\, S_0(k)$$
$$= 2\pi G R \int_0^\infty \mathrm{d}k\; J_1(kR)\, k \int_0^\infty \mathrm{d}R'\, J_0(kR')\, R'\, \Sigma(R')\,.$$

Example: Mestel disk

- Razor-thin disk with surface density

$$\Sigma(R) = \frac{v_c^2}{2\pi\, G\, R}$$

- Hankel transform:

$$\begin{aligned} S_0(k) &= \int_0^\infty \mathrm{d}R'\, J_0(kR')\, R'\, \frac{v_c^2}{2\pi\, G\, R'} \\ &= \frac{v_c^2}{2\pi\, G} \int_0^\infty \mathrm{d}R'\, J_0(kR') \\ &= \frac{v_c^2}{2\pi\, G\, k}\,, \end{aligned}$$

- Potential:

$$\begin{aligned} \Phi(R,z) &= -v_c^2 \int_0^\infty \mathrm{d}k\, \frac{e^{-k|z|}}{k}\, J_0(kR) \\ &= v_c^2\, \ln\!\left(\sqrt{R^2+z^2} + |z|\right). \end{aligned}$$

Example: Mestel disk

$$v_c^2(R) = v_c^2\, R \int_0^\infty \mathrm{d}k\, J_1(kR)$$
$$= v_c^2\,,$$

- Enclosed mass:

$$M(< R) = 2\pi \int_0^R \mathrm{d}R'\, R'\, \Sigma(R')$$
$$= \frac{v_c^2}{G} \int_0^R \mathrm{d}R'\, R'\, \frac{1}{R'}$$
$$= \frac{v_c^2\, R}{G}\,,$$

- Same as for spherical potential! Coincidence!

Example: Exponential disk

- Observed disks have exponential(-ish) surface density profiles:

$$\Sigma(R) = \Sigma(0)\, e^{-R/R_d}$$

- Hankel transform:

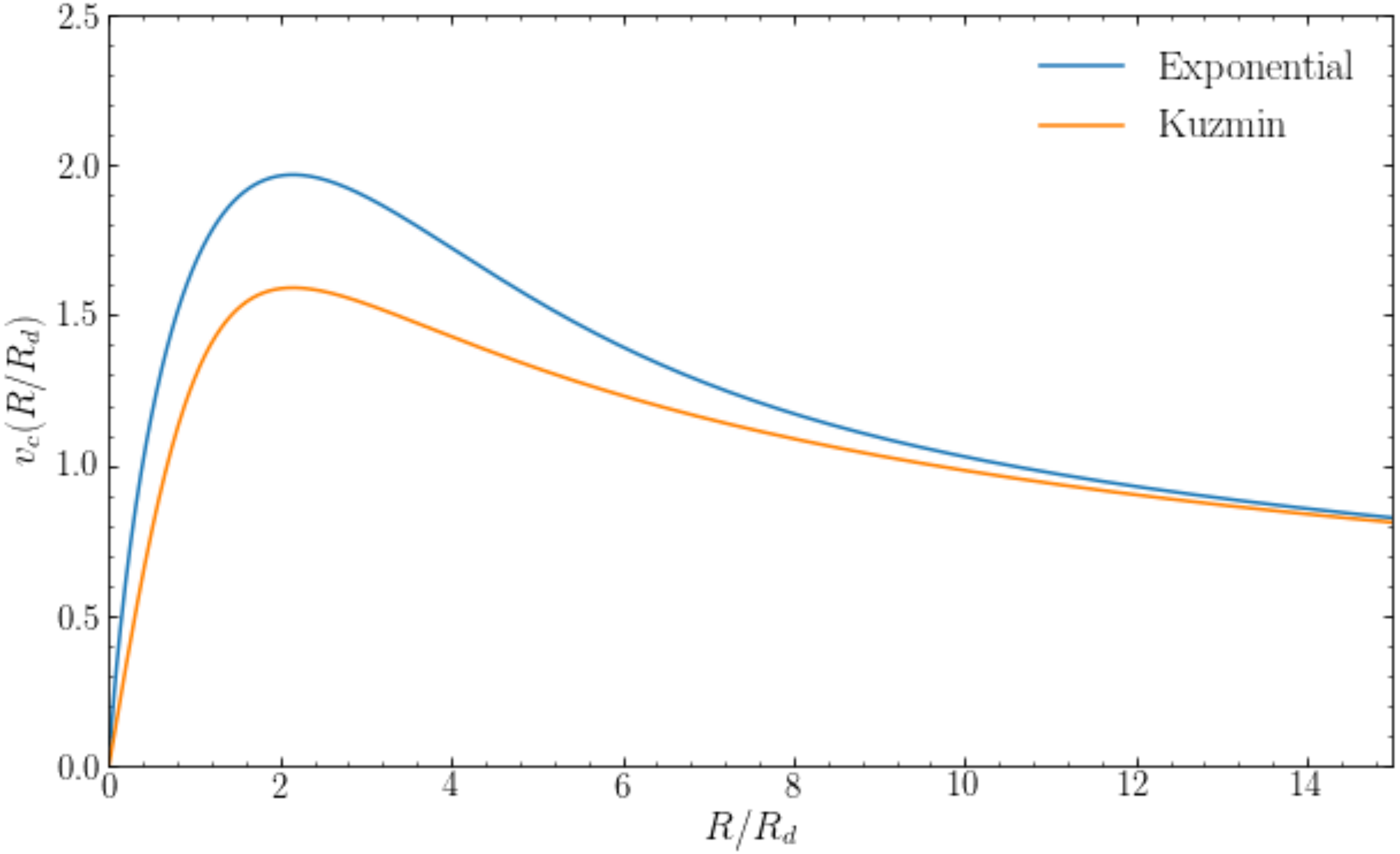
$$\begin{aligned} S_0(k) &= \Sigma(0) \int_0^\infty \mathrm{d}R'\, J_0(kR')\, R'\, e^{-\alpha R} \\ &= -\Sigma(0)\, \frac{\partial}{\partial \alpha} \int_0^\infty \mathrm{d}R'\, J_0(kR')\, e^{-\alpha R} \\ &= -\Sigma(0)\, \frac{\partial}{\partial \alpha} \left[(\alpha^2 + k^2)^{-1/2}\right]\,, \end{aligned}$$

- Potential:

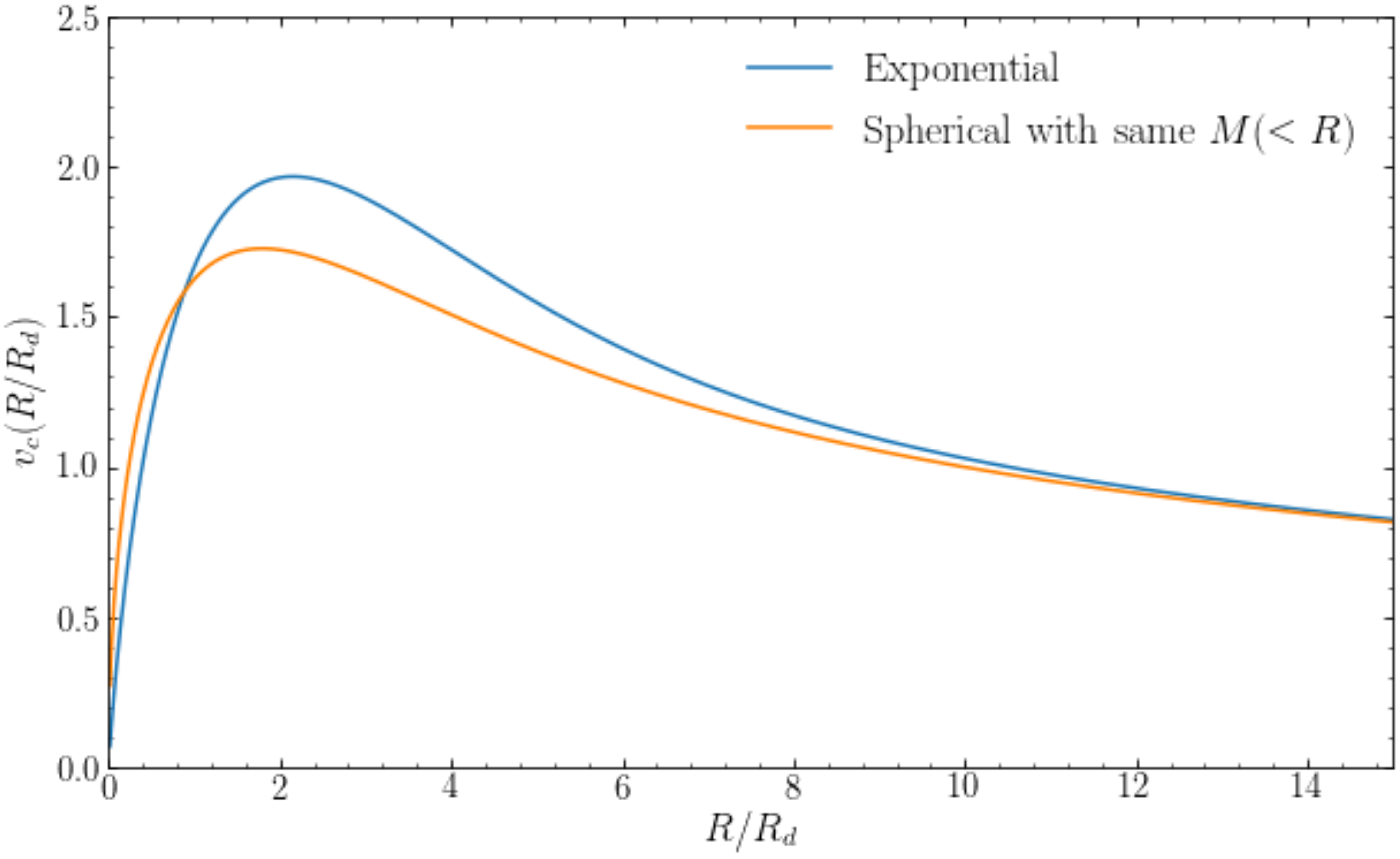
$$\begin{aligned} \Phi(R,0) &= 2\pi\, G\, \Sigma(0)\, \frac{\partial}{\partial \alpha} \int_0^\infty \mathrm{d}k\, J_0(kR)\, \left[(\alpha^2 + k^2)^{-1/2}\right] \\ &= 2\pi\, G\, \Sigma(0)\, \frac{\partial}{\partial \alpha} \left[I_0\left(\frac{\alpha R}{2}\right) K_0\left(\frac{\alpha R}{2}\right)\right] \\ &= \pi\, G\, \Sigma(0) \left[I_1\left(\frac{\alpha R}{2}\right) K_0\left(\frac{\alpha R}{2}\right) - I_0\left(\frac{\alpha R}{2}\right) K_1\left(\frac{\alpha R}{2}\right)\right] \end{aligned}$$

Example: Exponential disk

$$v_c^2(R) = \pi\, G\, \Sigma(0)\, \frac{R^2}{R_d} \left[I_0\left(\frac{R}{2R_d}\right) K_0\left(\frac{R}{2R_d}\right) - I_1\left(\frac{R}{2R_d}\right) K_1\left(\frac{R}{2R_d}\right) \right]$$



Example: Exponential disk



Potentials with finite thickness

- In general difficult, but galaxies overall well approximated as

$$\rho(R, \phi, z) = \Sigma(R, \phi)\, \zeta(z)\,,$$

- Then each layer gives rise to the ~same potential, but shifted in z

$$\mathrm{d}\Phi(R, z) = -2\pi\, G\, \mathrm{d}z'\, \zeta(z') \int_0^\infty \mathrm{d}k\, e^{-k|z-z'|}\, J_0(kR)\, S_0(k)\,.$$

- Full potential adds up contribution from each layer

$$\Phi(R, z) = -2\pi\, G \int \mathrm{d}z'\, \zeta(z') \int_0^\infty \mathrm{d}k\, e^{-k|z-z'|}\, J_0(kR)\, S_0(k)\,.$$

Example: *Double*-exponential disk

- Density for which both radial and vertical profile are exponential

$$\rho(R,z) = \Sigma(0)\, e^{-R/R_d}\, \frac{1}{2\,z_d}\, e^{-|z|/z_d} = \Sigma(0)\, e^{-\alpha R}\, \frac{\beta}{2}\, e^{-\beta|z|}$$

- Full potential + forces: one-dimensional numerical integrals

$$\begin{aligned}\Phi(R,0) &= -\pi\, G\,\beta \int \mathrm{d}z'\, e^{-\beta|z'|} \int_0^\infty \mathrm{d}k\, e^{-k|z'|}\, J_0(kR)\, S_0(k)\\ &= -\pi\, G\,\beta \int \mathrm{d}z' \int_0^\infty \mathrm{d}k\, e^{-(k+\beta)|z'|}\, J_0(kR)\, S_0(k)\\ &= -2\pi\, G\,\beta \int_0^\infty \mathrm{d}k\, \frac{1}{\beta+k}\, J_0(kR)\, S_0(k)\,.\end{aligned}$$

Double-exponential disk: rotation curve

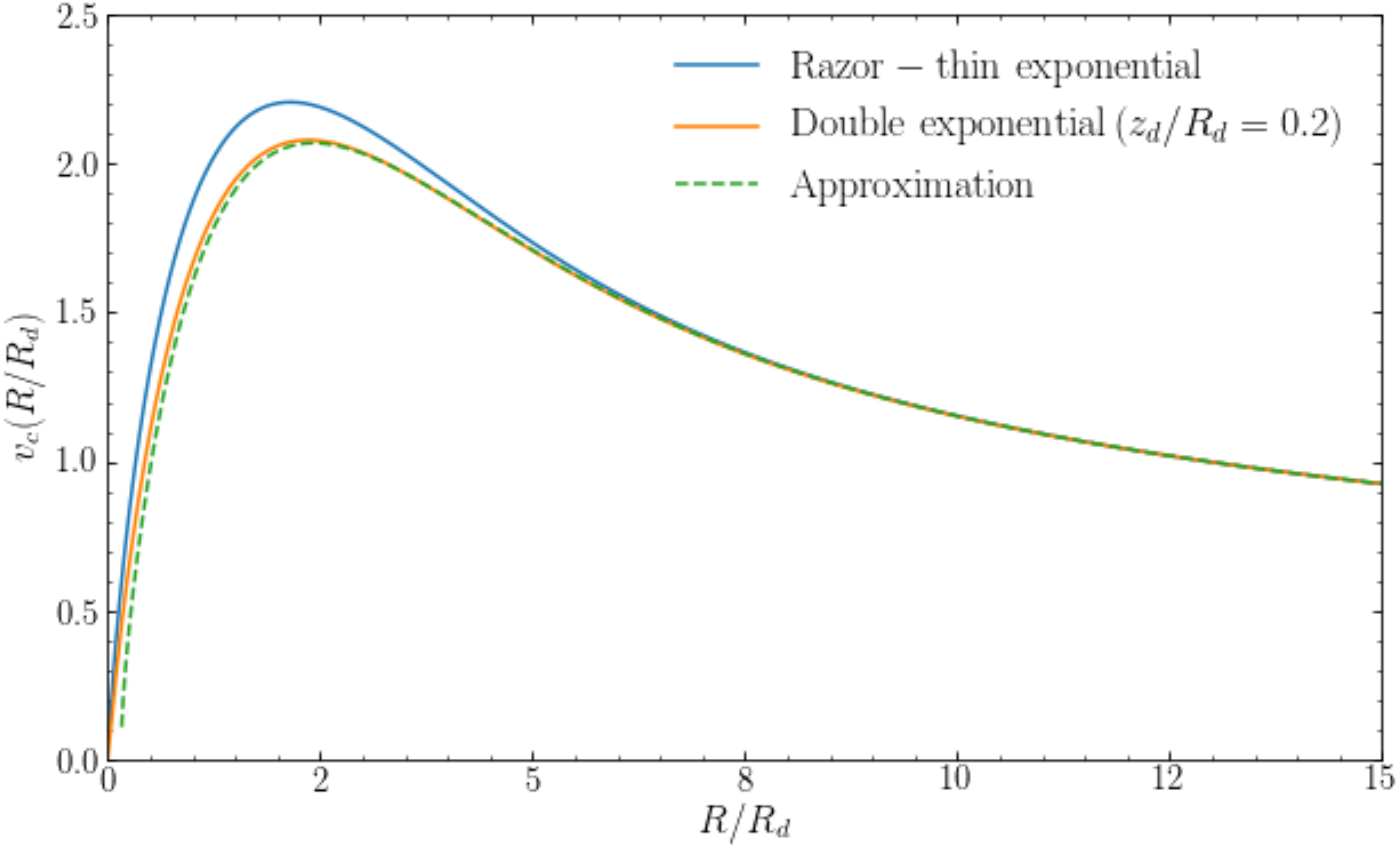
$$\Phi(R,0) \approx -2\pi G \int_0^\infty \mathrm{d}k \left(1 - \frac{k}{\beta}\right) J_0(kR)\, S_0(k)$$
$$\approx \Phi_{\mathrm{razor-thin}}(R,0) + \frac{2\pi G}{\beta} \int_0^\infty \mathrm{d}k\, k\, J_0(kR)\, S_0(k)$$
$$\approx \Phi_{\mathrm{razor-thin}}(R,0) + \frac{2\pi G}{\beta} \Sigma(R)\,,$$

- Rotation curve becomes:

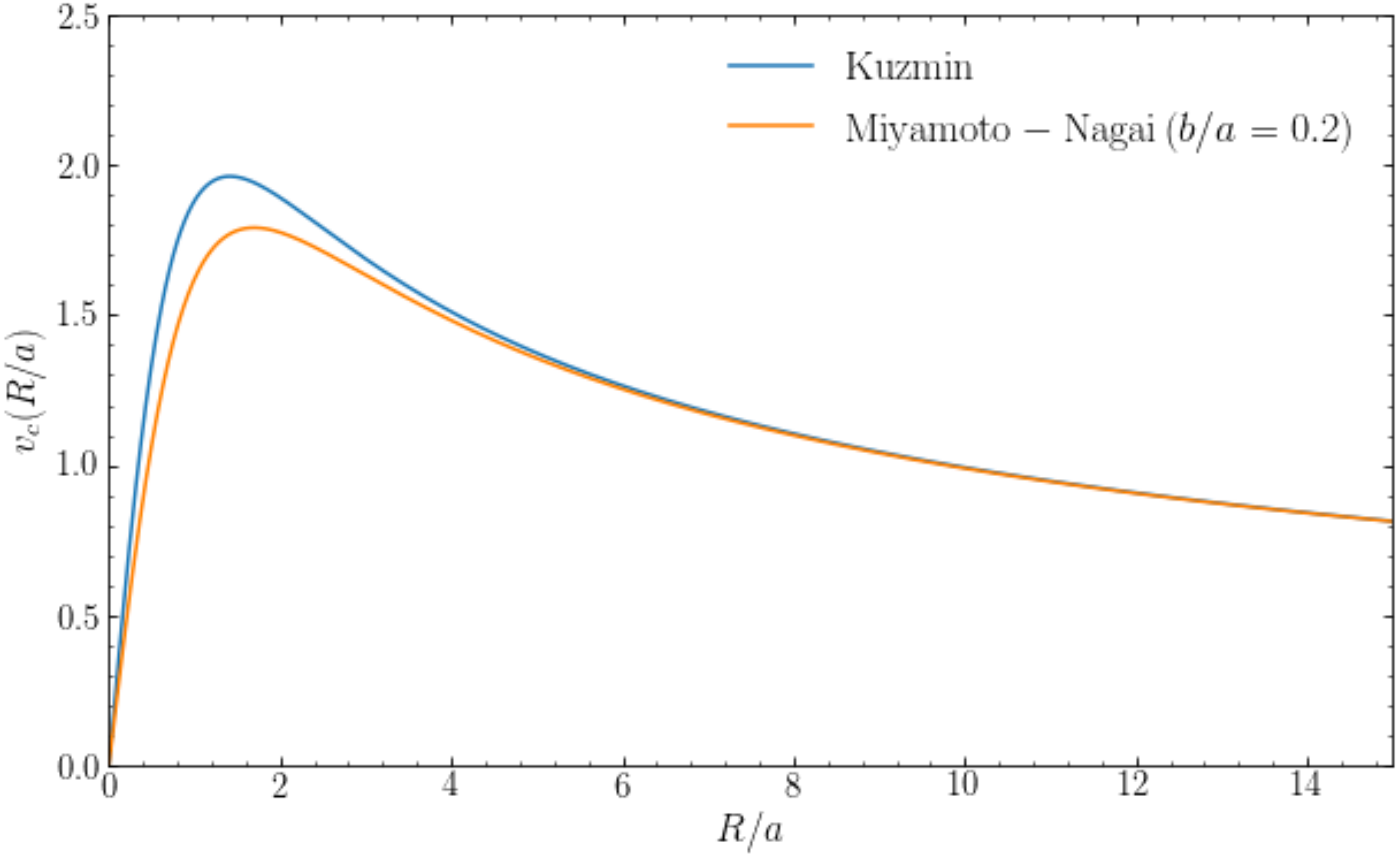
$$v_c^2(R) \approx v_{c,\mathrm{razor-thin}}^2(R) - 2\pi G\, \Sigma(0)\, \frac{z_d}{R_d}\, R\, e^{-R/R_d}\,,$$

$$\frac{\Delta v_c}{v_{c,\mathrm{razor-thin}}} \approx -\frac{5 z_d}{2 R_d}\, e^{-R/R_d} \approx 7\,\% \left(\frac{z_d/R_d}{0.2}\right)$$

Double-exponential disk: rotation curve



Miyamoto-Nagai rotation curve



Orbits in axisymmetric disks

Orbits in axisymmetric disks

- Approximate model for disk galaxy:
  - Flattened axisymmetric disk
  - Symmetric around z=0
  - Often use Miyamoto-Nagai for computational convenience
- E.g., *galpy*'s Milky-Way model
  - Miyamoto-Nagai disk with scale length 3 kpc, scale height 280 pc
  - NFW halo
  - Spherical bulge with exponential cut-off

Orbits in cylindrical geometry

- Lagrangian in cylindrical coordinates

$$\mathcal{L}(R, \phi, z, \dot{R}, \dot{\phi}, \dot{z}) = \frac{1}{2}\left(\dot{R}^2 + [R\,\dot{\phi}]^2 + \dot{z}^2\right) - \Phi(R, z)\,.$$

- With conjugate momenta

$$p_R = \frac{\partial \mathcal{L}}{\partial \dot{R}} = \dot{R}\,,$$
$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = R^2\,\dot{\phi}\,,$$
$$p_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = \dot{z}\,.$$

- Hamiltonian

$$H(R, \phi, z, p_R, p_\phi, p_z) = \frac{1}{2}\left(p_R^2 + \frac{p_\phi^2}{R^2} + p_z^2\right) + \Phi(R, z)\,.$$

Orbits in cylindrical geometry

- *z* component of the angular momentum is conserved

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = R^2 \dot{\phi} = \text{constant}$$

- Hamiltonian becomes

$$H_{\text{eff}}(R, z, p_R, p_z; L_z) = \frac{1}{2}\left(p_R^2 + p_z^2\right) + \Phi_{\text{eff}}(R, z; L_z) .$$

- with

$$\Phi_{\text{eff}}(R, z; L_z) = \Phi(R, z) + \frac{p_\phi^2}{2R^2}$$

- Effectively a two (four) dimensional system in (R,z) —> *meridional plane*

Motion in the meridional plane

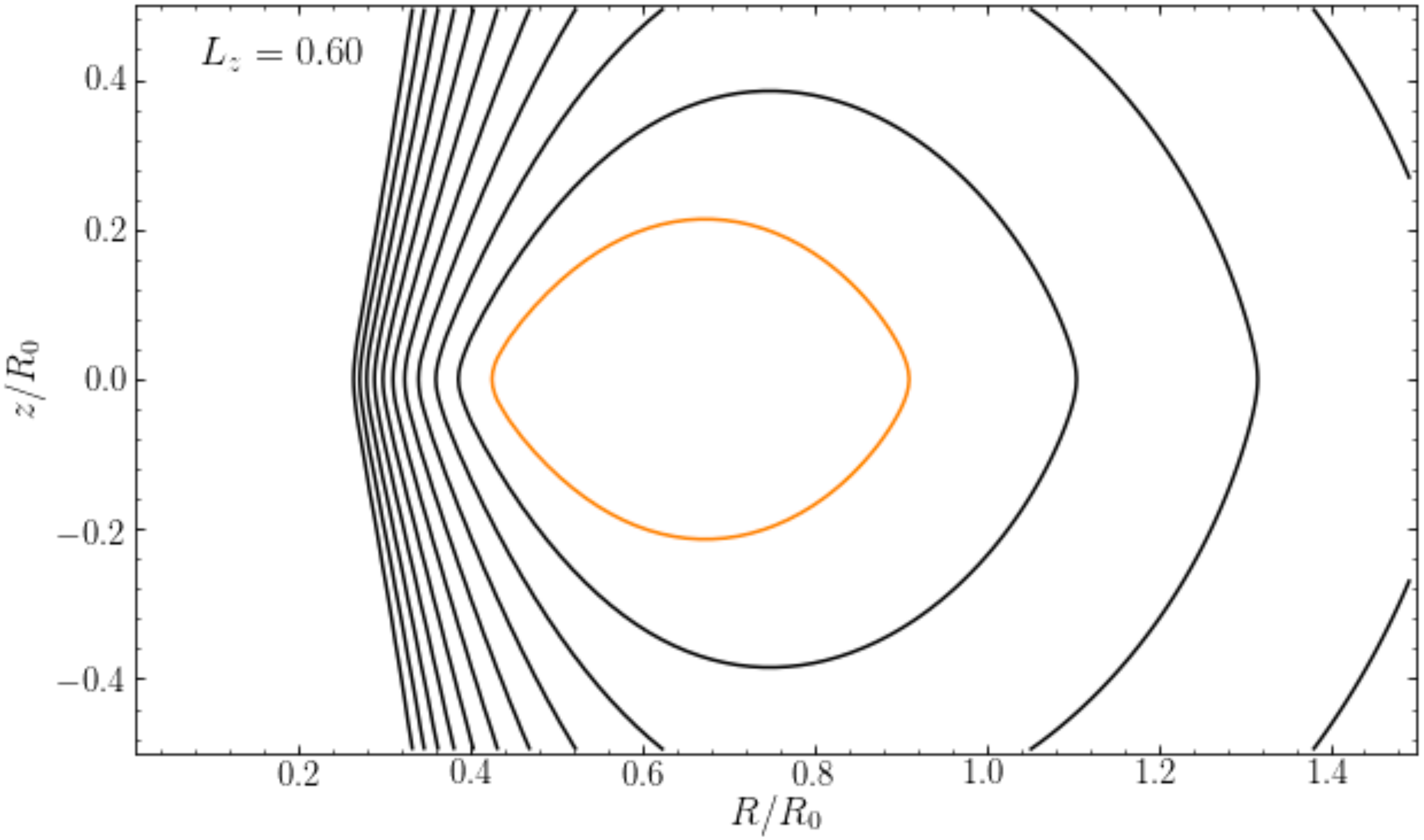
- Equations of motion

$$\dot{p}_R = \ddot{R} = -\frac{\partial \Phi_{\mathrm{eff}}(R, z; L_z)}{\partial R} = -\frac{\partial \Phi(R, z)}{\partial R} + \frac{L_z^2}{R^3};$$
$$\dot{p}_z = \ddot{z} = -\frac{\partial \Phi_{\mathrm{eff}}(R, z; L_z)}{\partial z} = -\frac{\partial \Phi(R, z)}{\partial z},$$

- Coupled oscillations in *R* and *z*

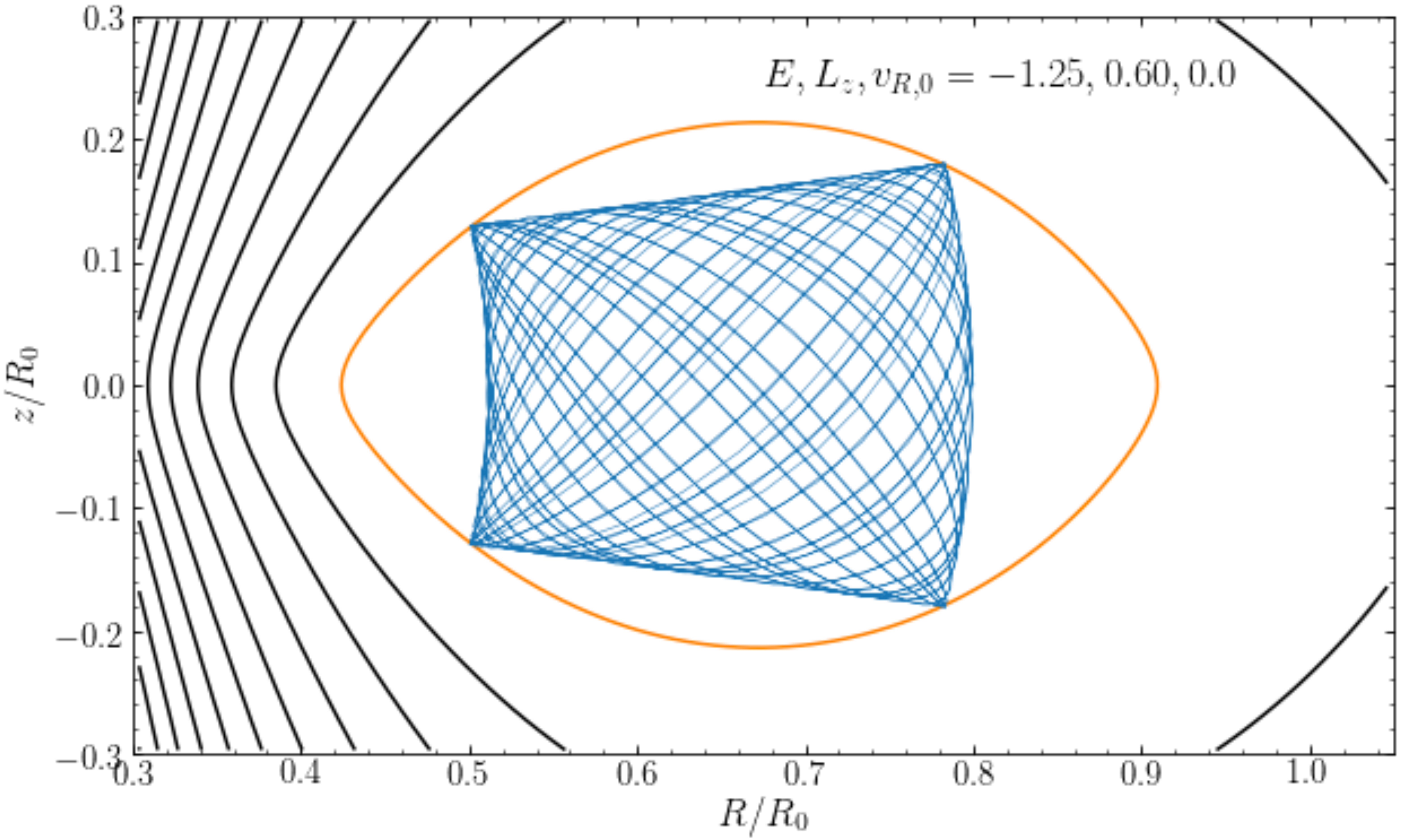
- No analytical solutions

Motion in the meridional plane



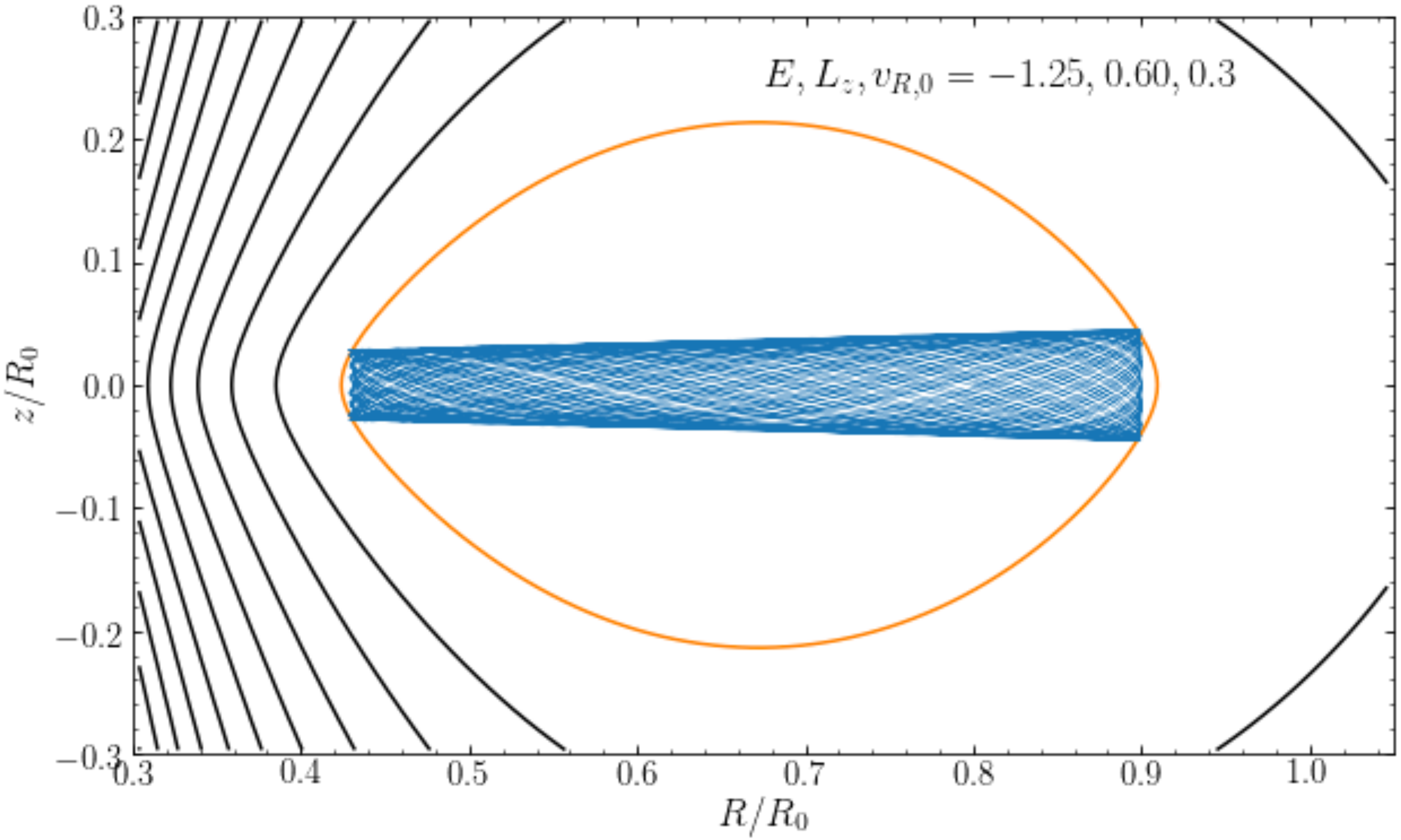
http://astro.utoronto.ca/~bovy/AST1420/orbits/lec5-orbitexample1.html

Motion in the meridional plane

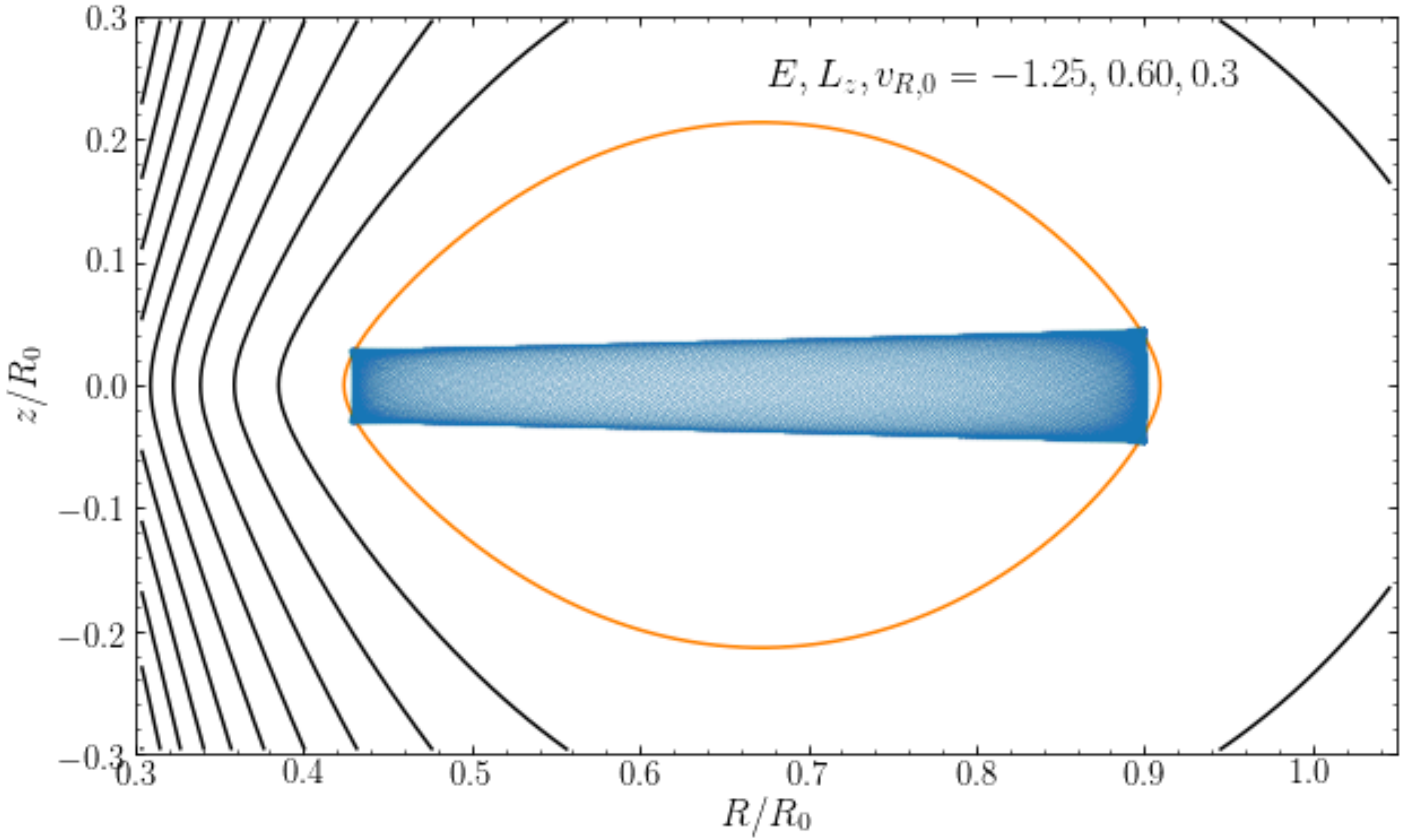


http://astro.utoronto.ca/~bovy/AST1420/orbits/lec5-orbitexample2.html

Motion in the meridional plane

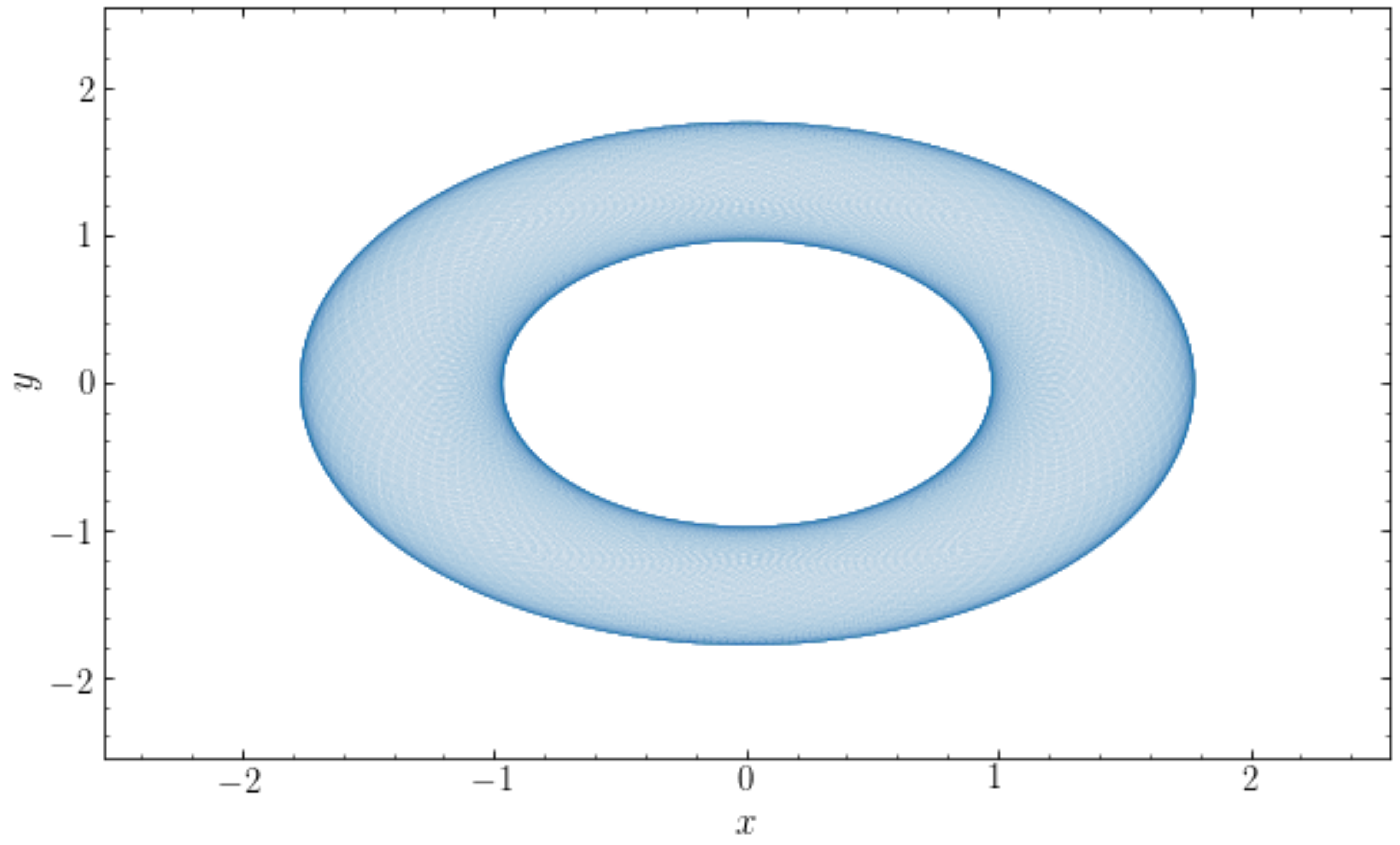


Motion in the meridional plane



Isolating integrals of motion

- Time-independent potential: E conserved —> motion restrained to ϕ < E
- Spherical potential: **L** conserved —> motion restrained to (a) orbital plane (b) vT = |**L**|/R (ϕ_{eff} < E)
- Motion fills rest of the phase-space



http://astro.utoronto.ca/~bovy/AST1420/orbits/lec5-orbitexample3.html

“Third” integral

- Fact that orbits in axisymmetric do not fully explore the area in the meridional plane alllowed by $\phi_{eff} < E$ means there has to be an additional integral
- No known classical integral (like E or L_z) —> non-classical ‘third’ integral
- Will come back to this in a few lectures