Problem set II, due Oct. 22nd, 9AM

In this problem set, we will study white dwarfs, which are a class of objects that are simple enough to be amenable to analytical tools, yet contain rich physics. Chapters 3, 4, 5, §7.2 §10.2 of HKV, as well as Chapter 35 of Kippenhahn & Weigert, provide useful background.

We will first construct white dwarf models assuming that they are zero temperature object. We will then give some heat to the ions and consider the cooling of the white dwarfs. Throughout, you will see how your results connect to observations. For sanity checks of your results, consult Hansen (1999, ApJ, 520, 680).

1 Hydrostatic Structure

1) assume electrons in the bulk of the white dwarfs are non-relativistically degenerate with zero temperature. Their equation of state is $P \propto \rho^{5/3}$, so these stars should be well described by a polytrope model with $n = 1.5$. What is the mass-radius relation for such objects? Table 7.1 of HKV gives numerical results for a few different polytrope models. In particular, we need the size for a $0.6 M_\odot$ white dwarf. (taking $\mu_e = 2$, as suitable for a helium or a carbon-oxygen white dwarf)

2) more massive white dwarfs are much denser so that their central degeneracy pressure can support against their larger self-gravity. When the electrons are so dense that they become relativistically degenerate, $P \propto \rho^{4/3}$ and the white dwarfs approach an $n = 3$ polytrope.[1] Relativistic white dwarfs have a single mass, the Chandrasekhar mass. Derive this mass and explain why, when white dwarf masses exceed this critical value, they will undergo collapse.

3) What is the lowest mass a white dwarf can have? Less massive white dwarfs have lower central densities and lower electron degeneracy pressure. However, a new source of pressure dominates over degeneracy pressure at low enough density, the Coulomb interaction. Ion electrostatic repulsion means that ions can be thought of as little hard spheres that can not be squeezed. Let the size of these spheres be the Bohr radius for the inner most electron surrounding a helium ion. The lowest mass white dwarf sits at the dividing line between Coulomb pressure and degeneracy pressure. Derive its mass. What is the mass-radius relation for objects smaller than this lower cut-off? By the way, they are called planets. How does your radius prediction for a 1 Jupiter-mass helium planet compare with that of the real Jupiter?

2 Cooling

Let us now return to study an average, $0.6 M_\odot$ carbon/oxygen white dwarf. We will from now on assume that the white dwarf has a temperature profile and is cooling gradually with time. Since a white dwarf is largely supported by degeneracy pressure, cooling does not lead to contraction. Energy source for the cooling is dominated by thermal energy of the ions.

4) The Mestel cooling curve. Define the white dwarf core to be where density is high enough for perfect degeneracy, and its envelope to be where density is so low that ideal gas equation of state operates. Let the density and temperature where the transition occurs be $\rho_{\rm env}$ and $T_{\rm env}$, respectively.

[1] An $n = 3$ polytrope has the peculiar property that it can have any radius for a given mass –it is interesting to figure out why for yourself.

- Write down an expression $\rho = \rho(T)$ at the core-envelope interface.
- Assume a cooling flux at the base of the non-degenerate envelope, F. Consider the envelope to be radiative and let its opacity be dominated by bound-free transitions (take $Z = 0.1$, $X = 0$). Solve for gas pressure, as a function of temperature, F, M and R, inside the envelope, by using the equation of hydrostatic equilibrium and radiative diffusion. You can safely assume that at the surface, the pressure and temperature are both much smaller than those at the core-envelope boundary, and that the mean molecular weight in the envelope, $\mu = 0.5$. You can also consider the envelope to be a very thin layer near the surface (a simplification to be justified below).
- Combine the above two results to obtain the density and temperature at the core-envelope interface, for given values of F and M. If the envelope is thin, the mass-radius relation derived in question 1 should remain valid.
- This should yield a relation between F and $T_{\rm env}$.
- The cooling of the white dwarf is very much controlled by how well heat can diffuse across its envelope. Its core, in contrast, poses virtually no resistance to heat diffusion. This is related to electron degeneracy. We can set the temperature from the center of the star to the base of the non-degenerate envelope to be a uniform T_c ($T_{\rm env} = T_c$). What is the total heat content of the material inside? if you are adopting some value for the specific heat, give an argument for why (e.g., c_v vs. c_p, degenerate vs. non-degenerate).
- Now write down an equation that relates the rate of cooling and T_c. Solving for this should give you a white dwarf cooling law, $L = 4\pi R^2 F = L(t)$.
- A particularly informative plot is to show how the central temperature T_c relates to the surface effective temperature ($T_{\rm eff}$) – you don't have to hand this in but I suggest you to try it yourself. The difference between the two is necessary for flux to diffuse through the radiative envelope.
- If you see a $0.6M_\odot$ white dwarf with a surface effective temperature of $12,000$ K, how old is it? What about a $1.2M_\odot$ white dwarf?[2]
- to justify our assumption that the non-degenerate layer is thin, obtain the pressure at the interface for the $0.6M_\odot$, $T_{\rm eff} = 12,000$ K white dwarf. This yields the column density of material above this layer, compare this value to that throughout the whole star.

5) if radiative opacity in the envelope is electron scattering instead, is the cooling accelerated or slowed down? you can argue qualitatively.

6) One important correction to the simple cooling curve arises from convection. A white dwarf can get so cool that part of its envelope can become convective. Let the envelope be made up of pure hydrogen (because helium and other heavy elements settle out quickly). In this case hydrogen ionization is responsible for destabilizing the atmosphere and gives rise to convection.

[2] This is exactly the method one uses to date white dwarfs, and the simple cooling curve we derive here is called the Mestel cooling curve (Mestel, 1952). Koester & Chanmugam (1990) demonstrated that more modern calculations agree roughly with this simple one. The oldest white dwarfs in M4, a halo globular cluster, have been dated to be 12 Gyrs old – they were formed when the universe was just 2 Gyrs old and the Milky way galaxy hasn't been assembled.

- Estimate at what surface temperature does convection first set in – use partial ionization as a marker. FYI, the photospheric pressure is expressed as

$$p = \frac{2}{3}\frac{g}{\kappa} \tag{1}$$

  where g is the gravitational acceleration and κ the opacity. Keep κ to be the bound-free opacity used above.

- As the white dwarf cools, the convection zone extends further and further inward. How does a deep convection zone affect the $T_c - T_{\rm eff}$ relation? and how does it impact the cooling?

- As the white dwarf cools past this point, its photospheric opacity decreases sharply (Fig. 1 of Hansen, 1999). How does this affect the $T_c - T_{\rm eff}$ curve and the cooling?

7) Another important change to the cooling curve occurs when the core temperature gets so cold that the core starts to become crystallized – ions are not degenerate at white dwarf densities, however, they may undergo a phase transition (into solid) when their kinetic energy is about a factor of 170 below the Coulomb energy between neighbouring ions. Estimate at what surface temperature this effect occurs for a $0.6M_\odot$ carbon-oxygen white dwarf. Ignore effects like electron screening around the naked ions.

Now you have understood white dwarfs inside out. In fact you are almost qualified to write an annual review article about white dwarfs – if only you were born 30 years earlier :-).