Contraction: Until ignition or ignominy *due 13 Feb 2023*

In this problem set, we will derive the minimum mass required to become a star powered by hydrogen fusion (rather than a brown dwarf). *Note: do not be overly put off by its length – it is easier than it perhaps looks. If stuck, do ask at office hours!*

Evolution of Central Properties

1. How does central temperature scale with central density as a star contracts during phases in which ideal-gas pressure dominates? Does contraction bring the star closer or further away from degeneracy?
2. Use your results to describe qualitatively what happens as a pre–main sequence star with a mass well below the Chandrasekhar mass contracts, ignoring nuclear processes.
3. What would happen for a pre–main sequence star with a mass well above the Chandrasekhar mass?

A rough estimate of the maximum temperature

1. A contracting low-mass pre–main sequence star is almost completely convective, and hence its interior structure is described well by a polytrope with index $n = 1.5$. Write out the central density and pressure in terms of mass and radius, and calculate numbers scaling mass and radius to solar units.
2. Assume degeneracy starts to dominate approximately when the pressure expected from a completely degenerate electron gas equals that expected from an ideal gas (and their sum equals the total pressure, i.e., $P_{\rm deg,c} = P_{\rm ideal,c} = \frac{1}{2}P_c$). Show that the temperature at that point, which we assume will be the maximum reached, is given by

   $$T_{\rm max} = 7.7\times10^7\,{\rm K}\,\mu\,\mu_{\rm e}^{5/3}\left(\frac{M}{M_\odot}\right)^{4/3}.$$
3. Given the above maximum temperature, it is a surprise the Sun is powered by fusion? Do you expect the gas in the Sun's core to be close to degenerate? To verify, look up the central density and temperature of the Sun and calculate the corresponding ideal-gas and degenerate-electron pressures.

Using a better approximation to the EoS

We will need maximum temperatures to estimate maximum fusion luminosities below. Since fusion is very sensitive to temperature, a precise estimate helps. In our estimate above, we approximated $P \simeq P_{\rm ideal} + P_{e,\rm deg}$, which is not very precise. A better approximation is given by (Paczynski, 1983, ApJ, 267, 315),

$$\begin{aligned}
P &= P_{\rm rad} + P_{\rm ion} + P_e\\
P_{\rm rad} &= \frac{1}{3}aT^4, \qquad P_{\rm ion} = \frac{\rho}{\mu_{\rm ion}m_H}kT, \qquad P_e = \left(P_{e,\rm ideal}^2 + P_{e,\rm deg}^2\right)^{1/2}\\
P_{e,\rm ideal} &= \frac{\rho}{\mu_e m_H}kT, \qquad P_{e,\rm deg} = \left(P_{e,\rm NRCD}^{-2} + P_{e,\rm ERCD}^{-2}\right)^{-1/2}\\
P_{e,\rm NRCD} &= K_1\left(\frac{\rho}{\mu_e m_H}\right)^{5/3}, \qquad P_{e,\rm ERCD} = K_2\left(\frac{\rho}{\mu_e m_H}\right)^{4/3}
\end{aligned}$$

For our application to low-mass stars, we can ignore both radiation and ERCD components, i.e., we have

$$P = P_{\rm ion} + \sqrt{P_{e,\rm ideal}^2 + P_{e,\rm NRCD}^2}.$$

Below, we will use that for given ratio of the electron degeneracy pressure to total pressure $x_{\rm deg} = P_{e,\rm deg}/P$ (and thus for given P and ρ), the ratio of ideal-gas pressure to total pressure $x_{\rm ideal} = (P_{\rm ion} + P_{e,\rm ideal})/P = (\rho/\mu m_H)kT/P$ is given by[1]

$$x_{\rm ideal} = \frac{1 - x_{\rm deg}^2}{1 - f_e + \sqrt{f_e^2 + (1 - 2f_e)x_{\rm deg}^2}} \quad \text{with} \quad f_e = \mu/\mu_e.$$

1. Use the above equation to calculate central temperatures for constracting stars of mass 0.03, 0.1, 0.3, and 1.0 $M_\odot$. Assume they can be described by $n = 1.5$ polytropes and have abundances $X = 0.7$, $Y = 0.28$ and $Z = 0.02$ (as in Fig. 2.3). Make a plot with curves of central temperature as a function of central density for all four stars. *(Here, easiest may be to take a range of radii, say from* 0.01 *to* $10\,R_\odot$*, calculate* $\rho_{\rm c}$ *and* $P_{\rm c}$ *for those, and then infer* $T_{\rm c}$ *only for radii for which* $P_{\rm e,deg} \leq P_{\rm c}$*.)*
2. Overdraw on the plot the central densities and temperatures for the three masses for which curves are shown in Fig. 2.3. Discuss whether the (mis)matches make sense (you also measured these in problem set 1; does what you find here confirm your previous conclusions?).
3. Use your solutions to calculate the maximum temperatures, as well as the radii and densities at which these are reached. Overplot on your figure, and verify that they follow,

$$\begin{aligned}
T_{\rm max} &= 9.8 \times 10^7\ {\rm K} \left(\frac{M}{M_\odot}\right)^{4/3} \\
R(T_{\rm max}) &= 0.052\ R_\odot \left(\frac{M}{M_\odot}\right)^{-1/3} \\
\rho(T_{\rm max}) &= 5.9 \times 10^4\ {\rm g\,cm^{-3}} \left(\frac{M}{M_\odot}\right)^{2}.
\end{aligned}$$

4. What are the fractions $x_{\rm deg} = P_{e,\rm deg}/P$ and $x_{\rm ideal} = (\rho/\mu m_H)kT/P$ for the maximum-temperature cases? Do those depend on mass? Why, or why not?
5. Also calculate the $T_{\rm c} - \rho_{\rm c}$ relation you would get if you made our previous, sloppy assumption that $P_{\rm ideal} = P - P_{e,\rm deg}$. Overdraw with a dotted line (for one mass is fine). Does the maximum match what you would expect from our first estimate of $T_{\rm max}$?
6. Consider a completely convective object just at the degenerate boundary. Will the outer layers be more, similarly, or less degenerate than the core? For this purpose, consider how $P_{\rm ideal}/P_{\rm deg}$ varies with radius. *Hint: If the answer seems far from obvious, consider first a place where the density has dropped by a factor two, and use scaling relations to calculate temperature, etc. If still stuck, check using the stellar models you construct below.*

[1] In case you are interested, you can derive as follows: write $P_{e,\rm ideal} = f_e P_{\rm ideal}$, $P_{\rm ion} = (1 - f_e)P_{\rm ideal}$, divide by P on both sides, take the ion term to the left, square both sides, and solve the resulting quadratic equation in $x_{\rm ideal}$ (using Muller's method instead of the usual quadratic formula to avoid getting a solution that has numeric problems at $f_e = 0.5$).

Luminosities

We now use `simple_star` to calculate luminosities for completely convective stars. For this purpose, use the p-p energy generation rate given by KWW, Eq. 18.63, taking $\psi = 1$ as appropriate for lower temperatures, and calculating the shielding correction $f_{1,1}$ using Eqs. 18.56 and 18.57.

1. We first revisit the cases of 0.1 and $0.3\,M_\odot$ stars from problem set 1 to check we can reproduce their properties. Here, start with assuming radii of 0.13 and $0.33\,R_\odot$, respectively, to estimate the central density and pressure, and use those to calculate the stellar structure. Verify that you get the correct radius. Next, use the equation for $x_{\rm ideal}$ above to infer the temperature at each radius. Verify that the central temperatures and densities are consistent with those of Fig. 2.3.
2. Use the structures you created above to calculate the hydrogen-burning luminosity L_r, by calculating the energy generation rate at each radius and summing appropriately.[2] You should find total luminosities $L \simeq 0.001$ and $0.007\,L_\odot$, respectively. Show that the implied effective temperatures are both be close to 2900 K (as is generally the case for stars at the Hayashi limit).
3. Plot M_r/M, $T/T_{\rm c}$, and L_r/L as a function of r/R. You should find that the mass and temperature distributions are identical for the two models, but that the luminosity is generated slightly closer to the centre for the $0.1\,M_\odot$ case. Why is this the case?
4. Now generate models for a range of masses, with radii chosen to be those that gives maximum central temperature. Calculate the generated luminosities for all your models and plot those as a function of mass.
5. Overdraw the radiated luminosity, assuming an effective temperature of 2900 K. At what mass does the maximum luminosity drops below the radiated one? Do you recover that the minimum mass for being a star is approximately $0.08\,M_\odot$?

[2] You could also add the energy balance equation to the structure equations that are integrated by `Polytrope`.