Problem set I. Polytropes for clouds and stars

due 30 Jan 2023

1. Stability of gas balls with an application to Barnard 68

Preparation (especially for questions 2 and 3): relevant sections in KWW.

1. Consider balls of ideal gas for which the virial theorem holds. Show that for a small inward or outward perturbation in radius, the ball is unstable if the gas is isothermal, while it is stable if no energy is lost (i.e., the ball reacts adiabatically, with $P \propto \rho^\gamma$ with $\gamma = 5/3$). What is the limiting value of γ?
2. Now consider a ball of gas that is not isolated, but embedded in an external medium. Follow the derivation of the virial theorem to show that for a ball of gas with a non-vanishing surface pressure P_s at its radius R, one has,

   $$E_\mathrm{pot} = -2E_\mathrm{kin} + 4\pi R^3 P_\mathrm{s}.$$

3. In equilibrium with an external medium with pressure P_ext, one must obviously have $P_\mathrm{s} = P_\mathrm{ext}$. How should P_s vary with radius for the equilibrium to be stable? Consider again a ball of isothermal gas and derive the resulting constraint on the radius in terms of M and T. *Hint: write* $E_\mathrm{pot} = -\Theta GM^2/R$ *and* $E_\mathrm{kin} = (M/\mu m_H)\frac{3}{2}kT$ *and assume fixed* Θ, M, μ *and* T.
4. We now turn to Barnard 68, a dark globule for which Alves et al. (2001, Nature 409, 159)[1] measured extinction as function of radius (see also Fig. 1.5). Look up the mass, radius, temperature, and external pressure they use/infer, make sensible estimates for Θ and μ, and check whether the cloud is indeed near equilibrium by calculating the three terms in the equation above. Also calculate the constraint on the radius required for stability, to see if you can confirm the paper's conclusion about the state of the cloud.
5. Given the external pressure and internal temperature of the Barnard 68, what must be the number density near the edge of the cloud? How does this compare with the average number density?
6. Download my simple-minded integrator for polytropic equations of state[2] (or write your own). Construct models for clouds for a range of central number densities, say from the average density to 20 times the average density. Plot the resulting profiles of P (log scale) and M_r (linear scale) as a function of radius. Give a qualitative explanation of your results.

[1] `https://www.nature.com/articles/35051509`

[2] `https://github.com/mhvk/simple_star`; this has installation instructions and an example as well.

2. Completely convective stars

Stars with masses $\lesssim 0.3\,M_\odot$ are completely convective, and a polytropic model should work fairly well. We will check this using the full stellar models shown in Fig. 2.3. Note: since we determine how well it works, keep a few significant digits in the calculations and read off the figure carefully (especially for density)!

1. Use the appropriate polytropic model to calculate central density and pressure for a completely convective $0.3\,M_\odot$ star with a $0.33\,R_\odot$ radius. Infer the central temperature, assuming an ideal gas with the composition indicated in Fig. 2.3. Repeat for a $0.1\,M_\odot$ star with $0.13\,R_\odot$ radius.
2. Compare your results with the central density and temperature shown in Fig. 2.3 for the two masses. You will find not all results are consistent. What assumption might be wrong? *(For the consistency check, assume a 10% uncertainty in radius – Pols et al. do not give radii and various theoretical models differ by this amount.)*
3. What is the expected slope in the temperature, density diagramme for $n = 1.5$? Make a copy of Fig. 2.3, and sketch a line with that slope starting at the centre for the $0.3\,M_\odot$ star. Does it match the straight part of the model curve? The wiggles further down are due to ionisation/dissociation zones; why does the track flatten in such zones?
4. For a $1\,M_\odot$ with $1\,R_\odot$ radius, the results should not match as closely. Is this true? What polytropic index n would work best? *(For the astute: you may notice that the central density in Fig. 2.3 is much lower than for the current Sun – this is because it is for a zero-age main-sequence star.)*
5. For the best estimate of n you found, sketch a line with the corresponding slope in the copy you made of Fig. 2.3, starting at the centre for the $1\,M_\odot$ star. Does it roughly match the model? Below $\log T \simeq 6.5$, the model track becomes much steeper, and runs parallel to the track for the $0.3\,M_\odot$ star. Which part of the Sun do you think this is?
6. Use the integrator for polytropic equations of state to calculate the internal structure of a star with the central density appropriate for the 0.3 and $1.0\,M_\odot$ star (using the appropriate γ and estimating K from the central density and temperature, assuming an ideal gas). Verify that the mass and radius make sense. Also, check that numbers do *not* make sense if you use the wrong γ. Finally, verify that $\rho_c/\overline{\rho}$ and $P_c/(GM^2/R^4)$ are as expected from Table 3.1.