Midterm AST320, 6 March 2015

Examination aids: Calculators only.

Note: The six parts have equal weight. Answers can be brief and in point-form, but be sure that derivations can be followed. A list of constants is appended.

1. Pre-main-sequence evolution.
   (a) Write down the Virial Theorem. Given this, how does the total energy $E_{\rm tot}$ of the star scale with mass M and radius R? As a pre-main-sequence star radiates, what happens to the potential, kinetic and total energy? (Assume ideal-gas pressure dominates.) How does this influence the radius, central temperature, $T_{\rm c}$ and central density, $\rho_{\rm c}$? Here, be sure to write down how $\rho_{\rm c}$ and $T_{\rm c}$ scale with M and R.
   (b) A $1\,M_\odot$ pre-main-sequence star has $\rho_{\rm c} \simeq 0.02\,{\rm g\,cm^{-3}}$ when it has $T_{\rm c} = 10^6\,{\rm K}$. Draw this point in the equation of state diagram on the next page, and sketch how $\rho_{\rm c}$ and $T_{\rm c}$ will evolve as the star contracts, indicating (roughly) where the contraction phase ends. Describe why the phase ends there.
   (c) Use scaling relations to estimate $\rho_{\rm c}$ at $T_{\rm c} = 10^6\,{\rm K}$ for stars with masses of 0.2 and $0.04\,M_\odot$. Draw these points in the figure as well, and sketch the evolution you would expect for each of these stars. For each, what eventually prevents it from contracting further?

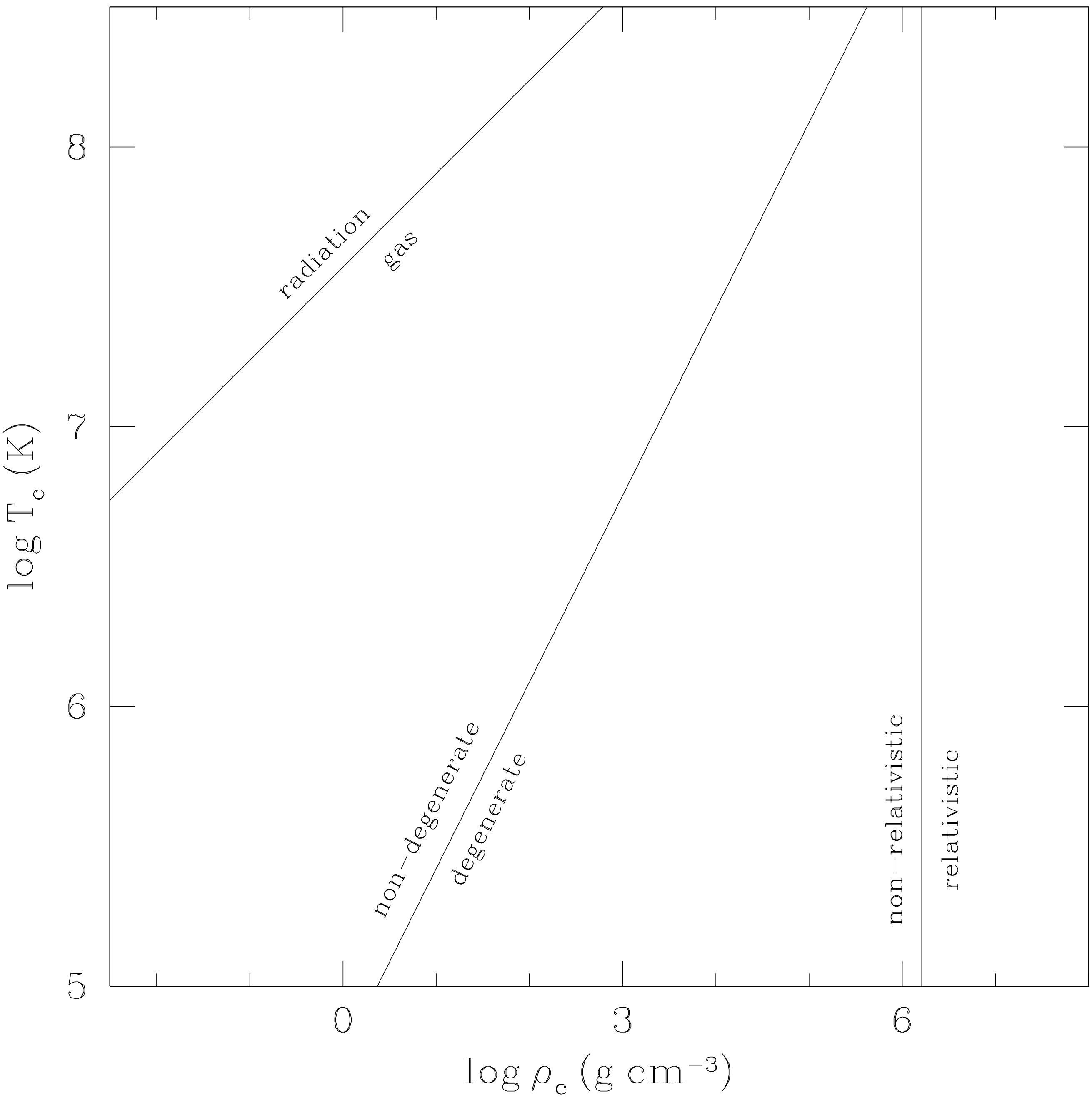

Fig. M.1. The T, ρ diagram for $X = 0.7$ and $Z = 0.02$, with the areas indicated where matter behaves as an ideal gas ($P \propto nT$), non-relativistic degenerate gas ($P \propto n_e^{5/3}$), relativistic degenerate gas ($P \propto n_e^{4/3}$), or radiation-dominated gas.

2. Nuclear fusion. For fusion of two nuclei a and b, the energy generation rate is generally,

$$\epsilon = [\text{some\#}\ \mathrm{J\,kg^{-1}\,s^{-1}}] f_{a,b} g_{a,b} X_a X_b \rho T^{-2/3} \mathrm{e}^{-19.721 (Z_a Z_b)^{2/3} (m'/m_{\mathrm{u}})^{1/3} (T/10^7\,\mathrm{K})^{-1/3}}. \tag{M.1}$$

Here, we consider primarily the exponential term.

(a) Explain qualitatively the physical reasons that changes in the various terms in the exponent (charges Z_a, Z_b; reduced mass m'; and temperature T) cause the energy generation rate to increase or decrease.

(b) Write down the six steps in the main (CN) part of the CNO cycle. What is the slowest step?

(c) Considering just the exponential term in Eq. M.1, how much slower would you expect this slowest step to be compared to the fusion of protons with protons (assume $T = 2 \times 10^7\,\mathrm{K}$)? Why is the CNO cycle relevant nevertheless (i.e., what physical effect compensates)?

Physical constants

(`http://physics.nist.gov/cuu/Constants`)

speed of light *in vacuo*	c	=	$2.99792458 \times 10^{8}\,\mathrm{m\,s^{-1}}$ (exact)
Gravitational constant	G	=	$6.673(10) \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
Planck's constant	h	=	$6.62606876(52) \times 10^{-34}\,\mathrm{J\,s}$
$[h/2\pi]$	$\hbar$	=	$1.054571596(82) \times 10^{-34}\,\mathrm{J\,s}$
Boltzmann's constant	k	=	$1.3806503(24) \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant			
$[\frac{1}{60}\pi^2 k^4/\hbar^3 c^2 = ac/4]$	σ	=	$5.670400(40) \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Avogadro's number	N_{A}	=	$6.02214199(47) \times 10^{23}\,\mathrm{mol^{-1}}$
Molar gas constant $[kN_{\mathrm{A}}]$	$\mathcal{R}$	=	$8.314472(15)\,\mathrm{J\,mol^{-1}\,K^{-1}}$
electron mass	m_{e}	=	$9.10938188(72) \times 10^{-31}\,\mathrm{kg}$
proton mass	m_{p}	=	$1.67262158(13) \times 10^{-27}\,\mathrm{kg}$

Other units

atomic mass unit $[\frac{1}{12}m(^{12}C)]$	m_{u}	=	$1.66053873(13) \times 10^{-27}\,\mathrm{kg}$
hydrogen mass	m_{H}	=	$1.6735525 \times 10^{-27}\,\mathrm{kg}$
electric charge	e	=	$1.602176462(63) \times 10^{-19}\,\mathrm{C}$
electron volt	eV	=	$1.602176462(63) \times 10^{-19}\,\mathrm{J}$
Ångstrom	Å	=	$10^{-10}\,\mathrm{m}$

Astronomical units

(Nautical Almanac 1993)

Solar mass	$M_\odot$	=	$1.9891 \times 10^{30}\,\mathrm{kg}$
Solar radius	$R_\odot$	=	$6.9551(3) \times 10^{8}\,\mathrm{m}$
Solar luminosity	$L_\odot$	=	$3.839(5) \times 10^{26}\,\mathrm{W}$ (not official)
Solar temperature	$T_{\mathrm{eff},\odot}$	=	$5777(2)\,\mathrm{K}$ (not official)
astronomical unit	AU	=	$1.49597870 \times 10^{11}\,\mathrm{m}$
parsec	pc	=	$3600 \times 180/\pi\,\mathrm{AU} = 3.0856776 \times 10^{16}\,\mathrm{m}$
Julian year	yr	=	$365.25 \times 84600\,\mathrm{s}$ ($\sim\pi\,10^{7}\,\mathrm{s}$)

Some formulae

ideal gas	$P = nkT$, $n = \rho/\mu m_{\mathrm{H}}$ $c_v = \frac{3}{2}Nk$, $c_p/c_v = 5/3$
non-relativistic degenerate gas	$P = K_1 n_{\mathrm{e}}^{5/3}$, $n_{\mathrm{e}} = \rho/\mu_{\mathrm{e}} m_{\mathrm{H}}$, $K_1 = \frac{1}{5}(3\pi^2)^{2/3}(\hbar^2/m_{\mathrm{e}})$ $K_1/m_{\mathrm{H}}^{5/3} = 9.91 \times 10^{6}$ (SI)
relativistic degenerate gas	$P = K_2 n_{\mathrm{e}}^{4/3}$, $n_{\mathrm{e}} = \rho/\mu_{\mathrm{e}} m_{\mathrm{H}}$, $K_2 = \frac{1}{4}(3\pi^2)^{1/3}\hbar c$ $K_2/m_{\mathrm{H}}^{4/3} = 1.231 \times 10^{10}$ (SI)
scale height	$\mathcal{H} = kT/\mu m_{\mathrm{H}} g$