Problem Set V: Accretion on a proto-Earth, Radioactive dating, Greenhouse effect

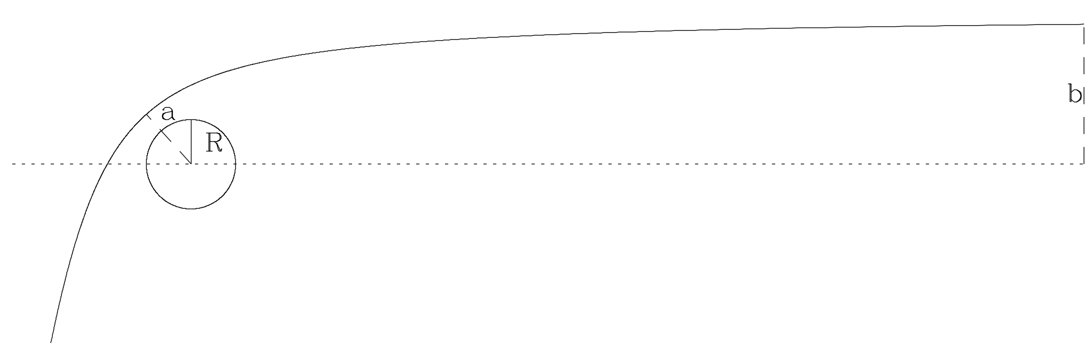
due 28 Nov 2008

For general comments about problem sets, see problem set I. For this specific one, you may find it useful to revisit CO, Ch. 2, on orbits, §20.4, “The Moon” (the part on radio-active dating), §20.3, “Earth” (greenhouse effect and global warning), and §9.4, “The Transfer Equation” (optical depth and Eddington approximation).

V.1. Accretion of planetesimals onto a proto-Earth

The figure below illustrates the trajectory of a planetesimal (rock of negligible mass and size) as it passes near a massive object (say, a proto-earth, of mass M and radius R). We are interested in the accretion cross-section for the latter object, and how its rate of growth depends on its mass.

1. We first consider the trajectory. Let the velocity of the rock relative to the proto Earth at infinity be v_∞ and the impact parameter (the separation between its initial trajectory and a parallel line through the centre of the proto-earth) be b. Gravity bends the trajectory producing a closest approach a ($a < b$). Obtain a as a function of b, M and v_∞. (*Hint: assume that the motion of the rock is only affected by the gravity of the proto-earth, and use that the rock's total energy and angular momentum are conserved.*)
2. The accretion cross-section $\sigma = \pi b^2$ is enhanced over the geometrical cross-section (πR^2) because of the gravitational focusing.
   (a) Set $a = R$ and derive σ. Write your results in terms of R, mean planet density ρ, and v_∞.
   (b) How large does the planet have to be for gravitational focussing to become significant? In other words, find the size $R = R_{\rm crit}$ (in terms of v_∞ and ρ) for which σ is enhanced over the geometrical value by a factor of two.
   (c) Calculate $R_{\rm crit}$ for $\rho = \rho_\oplus \simeq 5.5 \times 10^3\,{\rm kg\,m^{-3}}$ and $v_\infty = 1\,{\rm km\,s^{-1}}$ (a small fraction of the Keplerian velocity at 1 AU).
3. The proto-earth grows in mass by accreting planetesimals, at a rate $\dot{M}$ proportional to σ.
   (a) Write down σ in terms of ρ and M for the case that $R \gg R_{\rm crit}$.
   (b) Show that the time needed for a proto-planet to accrete its own mass, $t_{\rm acc} = M/\dot{M}$, scales as $M^{-1/3}$. (Thus, more massive objects grow faster; the 'rich get richer' scenario in planet formation.)



V.2. Radio-active dating

Radio-active rhenium (^{187}Re) decays with a half-life $\tau_{1/2} = 4.16 \times 10^{10}$ yr. The abundance for the decay product osmium (^{187}Os) rises accordingly so as to conserve the total number of nuclei of ^{187}Re and ^{187}Os. The abundances of these two elements are measured against the abundance of ^{188}Os, an isotope of Os which is not involved in any decaying process.

1. Show that the following equation is true:

$$\frac{N_{\rm Os}(t)}{N_{\rm Os2}} = (e^{\lambda t} - 1)\frac{N_{\rm Re}(t)}{N_{\rm Os2}} + \frac{N_{\rm Os}(0)}{N_{\rm Os2}} \tag{V.1}$$

   where $N_{\rm X}(y)$ is the number of nuclei of the element X at time y (Os2 for ^{188}Os), and $\lambda \equiv \ln 2/\tau_{1/2}$.

2. When a rock solidifies (from molten or vaporous forms), all elements are locked in, with initial ratios $N_{\rm Os}(0)/N_{\rm Os2}$ and $N_{\rm Os}(0)/N_{\rm Re}(0)$. Since different isotopes of the same element do not have different chemical behaviour, different minerals in the same piece of rock likely have the same $N_{\rm Os}(0)/N_{\rm Os2}$ value but differ in their $N_{\rm Os}(0)/N_{\rm Re}(0)$ values due to their different chemical compositions. At the present day, one can measure $N_{\rm Os}(t)/N_{\rm Os2}$ and $N_{\rm Re}(t)/N_{\rm Os2}$ for different minerals in the same rock. Show that $Y = N_{\rm Os}(t)/N_{\rm Os2}$ should depend linearly on $X = N_{\rm Re}(t)/N_{\rm Os2}$, with $Y = aX + b$, and write the values of a and b in terms of the other variables.
3. The table below lists a string of measurements for X and Y (taken from *Planetary Sciences*, de Pater & Lissauer). Use these to determine what is the initial abundance of $N_{\rm Os}(0)/N_{\rm Os2}$, and how long ago the rock solidified. (*Note: either use a least-square solver or use a graph to determine a and b. The measurements all have similar uncertainty.*)

$X = N_{\rm Re}(t)/N_{\rm Os2}$	$Y = N_{\rm Os}(t)/N_{\rm Os2}$
0.669	0.148
0.664	0.148
0.604	0.143
0.484	0.133
0.512	0.136
0.537	0.138
0.414	0.128
0.369	0.124

V.3. Greenhouse Effect and Global Warming

Perhaps the main worry of our time is global warming. Here, we make a overly simple model of the greenhouse effect to get an idea of how numbers scale. In class, we showed that for an air-less Earth, one can derive an equilibrium temperature $T_p = 255\,\mathrm{K}$ (also, Eq. 19.5 of CO), but this has to be modified when an atmosphere exists. Earth's atmosphere is optically thin (nearly transparent) at visible wavelengths so the solar radiation hits the ground directly. However, the atmosphere is optically thick (opaque) at infrared wavelengths and absorbs the ground's infrared black-body radiation. This heat is lost to space as the atmosphere radiates with a photospheric (top) temperature $T = T_p$ (think why; hint: energy conservation).

1. Imagine the atmosphere as a single opaque layer with a uniform temperature T_p. It is receiving heat from the ground (at temperature T_g) and radiates as much energy towards the ground as it radiates towards space. First ignoring the gradual warming of the atmosphere, use energy conservation to show that $T_g = 2^{1/4} T_p$. Is the current ground temperature (288 K) colder or hotter than this?
2. A more sophisticated approach is to allow different layers in the atmosphere to have different temperatures, each emitting both upwards and downwards, with a constant net flux passing through. From this, one can derive (CO, eq. 9.53) that the temperature will follow

   $$T^4 = T_p^4 \left[1 + \frac{3}{4}\left(\tau - \frac{2}{3}\right)\right]. \tag{V.2}$$

   where τ is the infrared optical depth from the point being considered to the top of the atmosphere. (As discussed in CO, the atmosphere emits at an effective optical depth $\tau = 2/3$.)
   (a) Given $T_g = 288\,\mathrm{K}$ and $T_p = 255\,\mathrm{K}$, what is the atmospheric optical depth τ_g to the ground?
   (b) Also calculate τ_g on Venus, given its no-atmosphere and actual ground temperatures.
   (c) Supposing, simplistically, that the greenhouse effect scales linearly with CO_2, what is the expected rise in temperature on Earth as CO_2 is doubled from the current abundance?
3. A more accurate prediction requires simulations which consider all greenhouse gases (e.g., water vapour has more effect than CO_2) and includes both positive and negative feedbacks as the earth's temperature rises. Look up the "climate change 2007 synthesis report" from the intergovernmental panel on climate change, and find what they predict for a doubling of CO_2.