Problem Set IV: Darwin instability *due 17 Nov 2017*

For general comments about problem sets, see problem set I. For this specific one, you may find it useful to read CO, §19.2, "Tidal forces.". Note that there are only two parts, but each is longer than typical for previous problem sets.

IV.1. Darwin instability: derivation

In class, we discussed how tidal friction will cause a very small body inside a planet's corotation radius to spiral in eventually. We will here consider a more general case. We start with two objects, a bigger one with mass M_1, and a smaller one, with mass M_2, where initially everything is in corotation, i.e., $P_{\rm orb} = P_1 = P_2$. As a measure of how difficult it is to change the rotation of object 1, we will write results in terms of its moment of intertia, I_1 (we will ignore the rotation of object 2, i.e., we assume $I_2 = 0$).

To test whether the corotation state is stable, we will consider a very slight deviation, in which the bigger object has spun down slightly and its spin period P_1 is thus ever so slightly longer than the orbital period $P_{\rm orb}$..

1. To get started, what are the rotational energy E_1 and angular momentum J_1 of object 1? What are the orbital energy $E_{\rm orb}$ and angular momentum $J_{\rm orb}$? Write all results in terms of M_1, M_2, I_1, P_1 and $P_{\rm orb}$.
2. In general, for tidal interaction, $J_1 + J_{\rm orb}$ is conserved, but $E_1 + E_{\rm orb}$ is not. What other source or sink of energy is present?
3. Now consider our starting situation, with object 1 rotating slightly slower than the orbital period.
   (a) In which direction will energy and angular momentum flow, from the orbit to the rotation of 1, or vice versa? Illustrate your answer with a sketch of the situation, indicating tidal bulges, forces, etc.
   (b) Supposing a new equilibrium can be reached, will object 2 be further or less far away from object 1? And will the rotation of object 1 be faster or slower than before?
4. To determine whether a new equilibrium can in fact be reached, show
   (a) That conservation of angular momentum implies $J_1(\dot{P}_1/P_1) = \frac{1}{3}J_{\rm orb}(\dot{P}_{\rm orb}/P_{\rm orb})$;
   (b) That, for stability, you thus require $J_{\rm orb} > 3J_1$.

IV.2. Darwin instability: application

We now apply the Darwin instability criterion. In order to calculate the angular momentum of the spinning object 1, we will express its moment of intertia as $I_1 = kM_1R_1^2$, with k a constant; you can use that $k \simeq 0.33$ for Earth, 0.26 for Jupiter, and 0.06 for the Sun.

1. We first consider k.
   (a) What is k for a constant-density sphere?
   (b) Explain qualitively why this is larger than the values for Earth, Jupiter, and the Sun.
   (c) Explain qualitatively the relative ranking of Earth, Jupiter, and the Sun, based on the properties of their interiors.
2. Apply the stability criterion to the X-ray binary Cen X-3, which is composed of a massive, $20\,M_\odot$, $12\,R_\odot$ star, orbited by a $1.4\,M_\odot$ neutron star ($P_{\rm orb} = 2.087\,$d; pick a reasonable k, and assume $P_1 \simeq P_{\rm orb}$). Do you think astronomers were surprised when they found the orbit was decaying on a 10^5 yr timescale?
3. Now we consider Mars (again pick a reasonable k, and assume $P_{\rm orb} \simeq P_{\rm Mars}$).
   (a) Derive a general expression for the minimum mass M_2 required for stability for the case that $M_2 \ll M_1$.
   (b) Now insert numbers for Mars. How does this compare with Phobos?
4. Many Hot Jupiters, with masses of $\sim 1\,M_J$, are in ~ 4 day orbits with their Sun-like hosts.
   (a) Are these systems Darwin-unstable? (*Feel free to use the relation you derived for 3a.*).
   (b) Does your answer leave you surprised about their existence? If so, what could explain the apparent discrepancy?