# Physical Ingredients for Constructing a Star (or Planet)

1. Support against gravity
   - Pressure: hydrostatic equilibrium
     equation of state
   - Energy: virial theorem
2. Source of energy
   - Contraction
   - Nuclear power
   - Photon propagation/emission

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# Nuclear Fusion

Einstein: $E = m c^2$

atomic unit: $u = 1.660540 \times 10^{-27}$ kg ( mass of $^{12}C/12$)

proton: $m_p = 1.672623 \times 10^{-27}$ kg $= 1.0072765\ u$ = **938.79 MeV**

neutron: $m_n = 1.674929 \times 10^{-27}$ kg $= 1.0086653\ u$

electron: $m_e = 9.109390 \times 10^{-31}$ kg $= 0.0054858\ u$
Hydrogen atom: $m_H = 1.007825\ u = m_p + m_e -$ electro-static/2 → 13.6 eV binding energy of H atom

Helium nucleus: $m_{He} = 4.00151\ u = 2\ m_p + 2\ m_n - \Delta m$
$\Delta m = 0.03037\ u \sim 0.7\%\ (4\ m_H) \sim 28$ MeV

Fusion 4 p → He releases ~28 MeV

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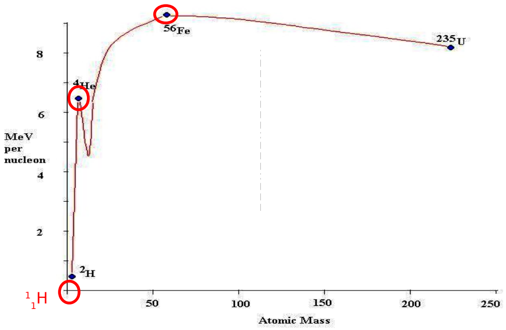
## Binding Energy of a Nucleus

*--- energy released when formed*
*--- energy required to unbind*

| | | |
|---|---|---|
| $^1_1H$ | $E_b = 0$ | |
| $^4_2He$ | $\Delta m c^2 = 28.03$ MeV, hence | 7.08 MeV/nucleon |
| $^{16}_8O$ | $(Z m_p + (A-Z) m_n - m_{nucleus}) c^2/A=$ | 7.97 MeV/nucleon |
| $^{56}_{26}Fe$ | | 8.79 MeV/nucleon |
| ($^{238}_{92}U$ | | 7.3 MeV/nucleon) |

Most stable nucleus easily made in stars.



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# Basic concepts in nuclear physics

1) Binding energy of an element: $^A_ZX$: $E_b = (Z\ m_p + (A-Z)\ m_n - m_{nucleus})\ c^2$

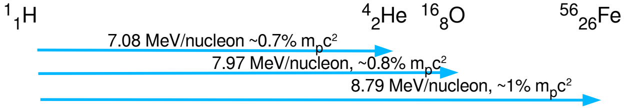
2) attractive **strong force** binds protons & neutrons together against repulsive Coulomb force

*Four basic forces in nature*
long-range forces: gravity, electro-magnetism (force carriers have no mass, F ~1/r²)
short-range forces: strong force, weak force (force carries have mass, strength falls off dramatically beyond nuclear dimension ~1 fm = $10^{-15}$ m)

3) For *A* less than a critical value (~56),
strong force increases with *A* faster than Coulomb repulsion
⇒ binding/nucleon increases with *A*
Beyond that, nucleus too large ⇒ binding/nucleon decreases *A*

---- energetically favourable to fuse H to He ..... to Fe, but not past Fe
---- past Fe, fission energetically favourable; fusion becomes endothermic

## Nuclear Energy Yield: H burning to Fe yields ~ 9 MeV/nucleon



Fusion: Each proton can maximally yield ~1% $m_pc^2$ ~ 9 MeV ~ $10^{-12}$ J

*(1 g of Hydrogen fusion ~ annual energy consumption of one Canadian)*

Fusion energy available for Sun: $E \sim 10^{-12}$ J x $M_\odot/m_p \sim 10^{45}$ J >> $GM_\odot^2/R_\odot$

**Nuclear Timescale:** $t_{nuc} \sim E/L_\odot \sim 10^{45}$ J/$L_\odot \sim 10^{11}$ yr

Actual lifespan ~$10^{10}$ years (now about half-way)
1) *L* increases in later life;
2) not all H burned;
3) not burned to Fe.

Compare: dynamical time: $t_{dyn} \sim$ 30 min
thermal time: $t_{KH} \sim 10^7$ yr

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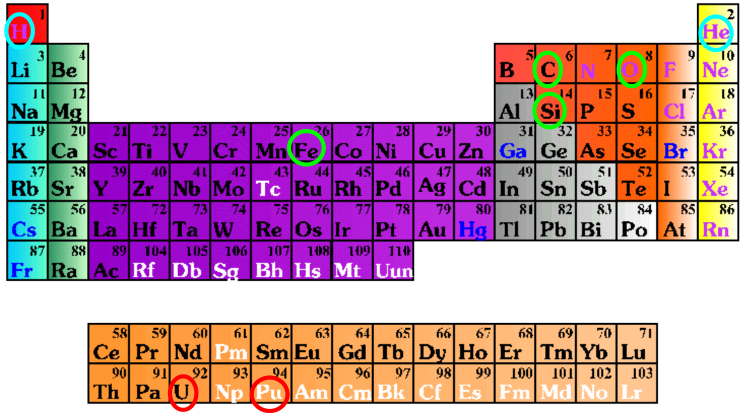
Reminder: Periodic Table of the Elements (from http://www.chemicool.com)



*Uranium & Plutonium*

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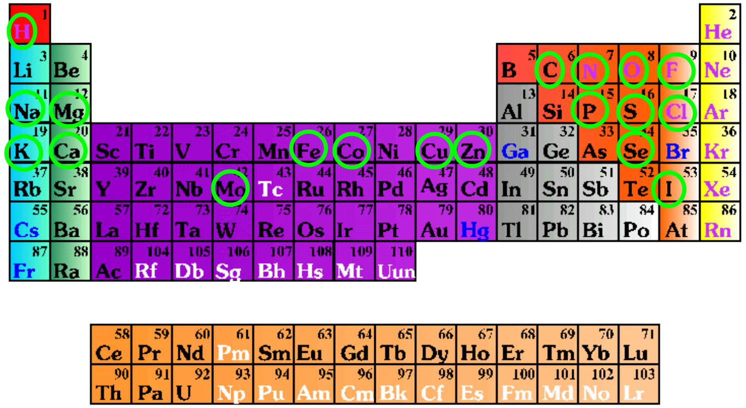
Elements used by the Human body



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## How to set the Nuclear Fire?

*(or why we don't yet have clean fusion power on Earth)*

4 $^1_1H \rightarrow {}^4_2He$ + energy release of ~7 MeV/nucleon

Problem 1: **Coulomb barrier**

protons have to **overcome the electrostatic repulsion** between them and reach the realm of strong force (~1 fm = $10^{-15}$ m, also the size of a nucleus)

*($e^-$ too far out to be relevant, ~$10^{-11}$ m)*

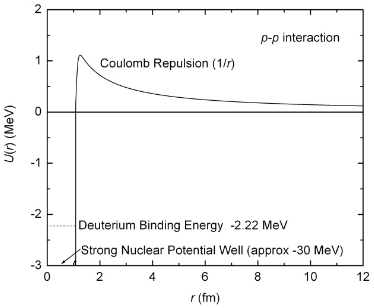
## Classical solution: **thermal motion of the nuclei**

*Two protons need to get as close as ~1 fm for strong force attraction to set in*

*Coulomb Force is a potential force (like gravity, but can repel)*

*Coulomb potential* $U_C = \frac{1}{4\pi\epsilon_0}\frac{e_1 e_2}{r}$ *(1.4 MeV for* $r = 1\,fm$*)*

*As two protons approach each other, total energy is conserved:*



$$\frac{1}{2}m_p v_\infty^2 + U_c(\infty) = U_c(1\,fm)$$

$$\frac{1}{2}m_p v_\infty^2 = \frac{3}{2}kT \geq \frac{e^2}{4\pi\epsilon_0(1\,fm)}$$

$$T \geq T_{classical} = \frac{e^2}{6k\pi\epsilon_0(1\,fm)} \sim 10^{10}\,K$$

But from Virial Theorem,
we know that Sun has $T_c \sim 10^7$ K

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Sir Arthur Eddington (1882-1944)

Arthur Eddington thought that nuclear processes must be involved to account for the radiant energy of the sun, but was criticized because the temperature was seen to be not hot enough when considered by classical physics alone.

His tongue-in-cheek reply to his critics: "*I am aware that many critics consider the stars are not hot enough. The critics lay themselves open to an obvious retort; we tell them to go and find a hotter place.*"

Vindicated by quantum mechanics.
Protons are 'BIG'.

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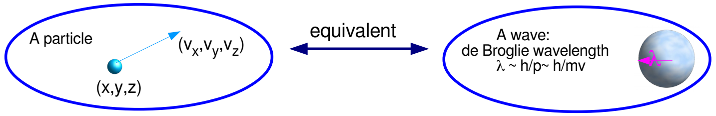
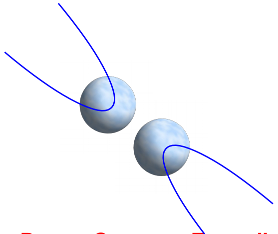
## Quantum mechanical solution: **Wave-Particle Duality**

**(size of a particle depends on its momentum)**



Use *Quantum Tunneling:* if a proton gets within one de Broglie wavelength of another, there is a certain probability that they "are" closer than ~1 fm



$$U_c = \frac{e^2}{4\pi\epsilon_0\lambda} \leq \frac{1}{2}m_p v_\infty^2 = \frac{1}{2}\frac{p^2}{m_p} \sim \frac{1}{2m_p}\left(\frac{h}{\lambda}\right)^2$$

$$\lambda \leq \frac{4\pi\epsilon_0 h^2}{2e^2 m_p} \sim 10^{-13}\,m \sim 100\,fm$$

$$\frac{3}{2}kT = \frac{1}{2}m_p v_\infty^2 = \frac{1}{2m_p}\left(\frac{h}{\lambda}\right)^2$$

$$T \geq T_{quantum} \sim \frac{m_p e^4}{12\pi^2\epsilon_0^2 k h^2} \sim 10^7\,K$$

**Due to Quantum Tunneling, hydrogen fusion is possible already at $10^7$ K**

*(stars < $0.08M_\odot$ cannot reach this temperature at the center; "failed stars" or brown dwarfs)*

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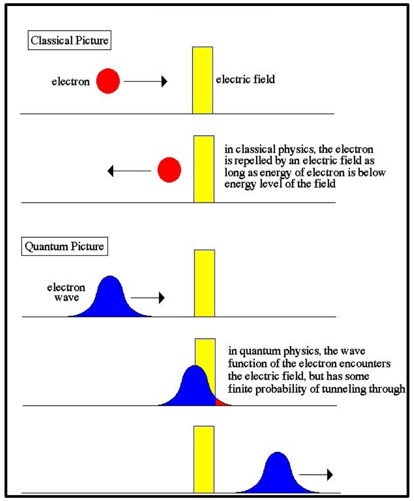
## Quantum tunneling and the ignition temperature



in classical physics, the electron is repelled by an electric field as long as energy of electron is below energy level of the field

in quantum physics, the wave function of the electron encounters the electric field, but has some finite probability of tunneling through

Quantum Tunneling:
a physical system spends some time in an energetically forbidden region:

*there always exists*
*a certain probability*
*that two protons 'are'*
*closer than ~1fm*

***Tunneling probability***

$\exp(-2\pi^2 U_c(\lambda(E))/E)$

$\sim 4\times10^{-11}$ $(E/k \sim T \sim 10^7\,K)$
$\sim 0.4$ $(E/k \sim T \sim 10^{10}\,K)$

Nuclear reaction rates rise extremely steeply with T.
Hence, **ignition temperature**
($\sim 10^7$ K for Hydrogen fusion)

# How to set the Nuclear Fire? (cont'd)

**Problem 2: How often do protons see each other?**

1) space is empty for nuclei even at the Sun's center ($\rho \sim 100$ g/cm$^3$)
   mean separation $\sim 10^{-11}$ m $\gg \lambda \sim 10^{-13}$ m

2) $4\ {}^1_1\mathrm{H} \rightarrow {}^4_2\mathrm{He}$ + energy release of ~7 MeV/nucleon
   All four in the same place at the same time? --- difficult
   Fusion proceeds through *a chain of 2-body reactions,*
   each satisfying conservation of energy, momentum, charge,
   lepton number ($e^-$, $e^+$, $\nu$, anti-$\nu$), and baryon number (A)

   a) At low T: **p-p chain** (involving only protons and products);
   b) At higher T: **CNO chain (**C,N,O as catalysts).

3) The more highly charged the nucleus, the higher the Coulomb barrier,
   and thus the higher the required temperature;
   Helium fusion requires $T > 10^8$ K; others $T > 10^9$ K
   *(relevant after H exhaustion in the stellar core)*