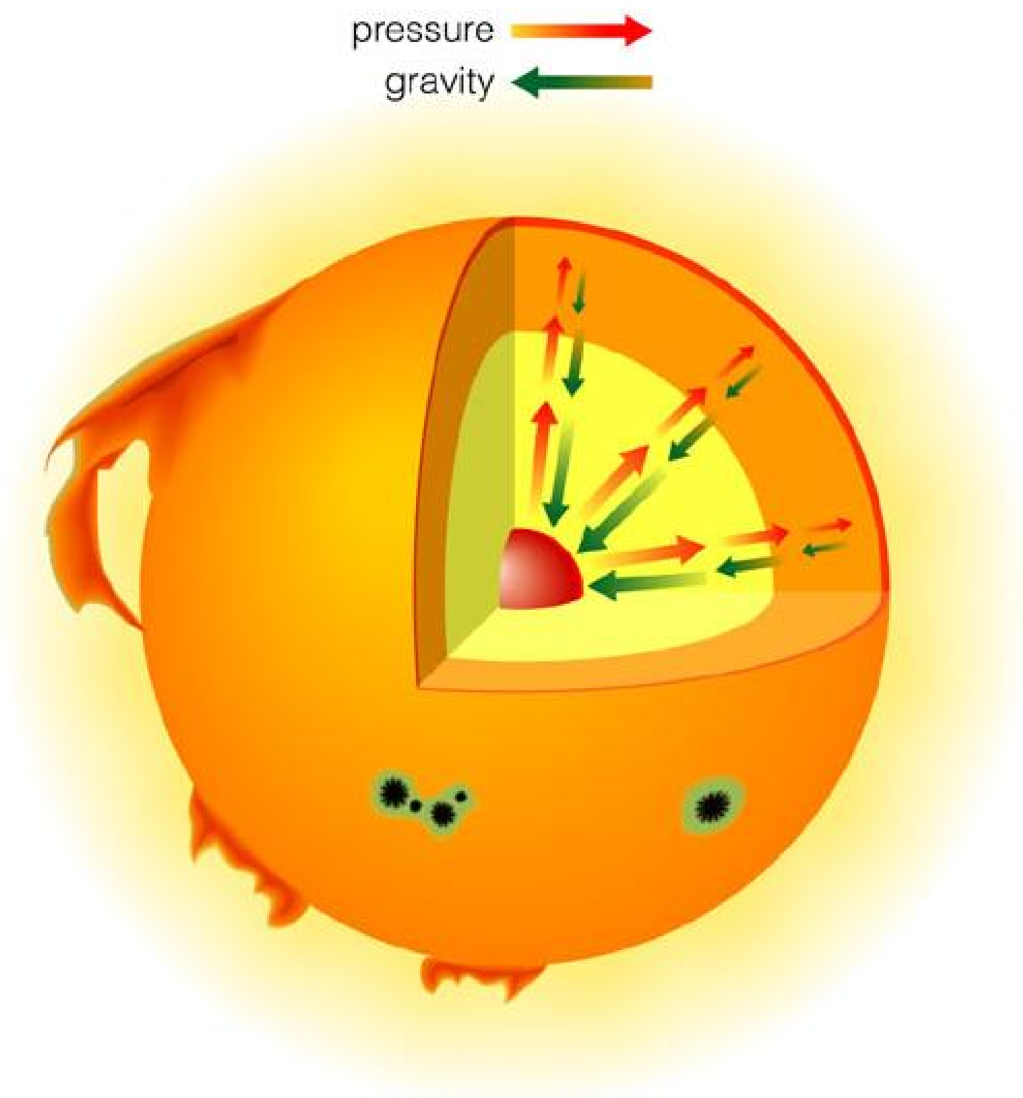
Star's life: Protracted battle with gravity



ALWAYS

To support weight:

⇒ need high pressure

MOSTLY

⇒ need high temperature

⇒ will loose energy

⇒ need energy source:
- Gravitational contraction
- Nuclear fusion

Ultimately,
Can something else than thermal pressure balance gravity?

Virial Theorem

(Claussius 1870)

For a bound gravitational system, the long-term average of its kinetic energy is one half that of its potential energy.

Kinetic Energy: 1) orbital kinetic energy (bulk motion)

2) thermal kinetic energy (random motion)

only 1) rotational support against gravity
Earth-Moon, Earth-Sun, the Milky Way galaxy (spiral galaxies),....

only 2) “pressure” support against gravity
stars & gaseous planets, globular clusters of stars

1+2) mixed support
elliptical galaxies, super-cluster of galaxies

Virial Theorem: <1> + <2> = - ½ <U>

(total energy = <1 + 2 + U > = ½ <U> [less than 0])

Proof of the Virial Theorem

For a ball of gas in hydrostatic equilibrium)

Gravitational Energy: $U = -\int_0^R \frac{G M_r \rho}{r} 4\pi r^2 \, dr = -f \times \frac{G M^2}{R}$

$f = \frac{3}{5}$ *for uniform density,* $f > \frac{3}{5}$ *if star is centrally concentrated.*

Kinetic Energy: each gas particle has thermal kinetic energy of 3/2 $k_B T$

$$K = TE = \int \frac{3}{2} k_B T \, n \, dV = \frac{3}{2} \int_0^R P \, 4\pi r^2 \, dr = \left[\frac{6\pi}{3} P r^3 \right]_0^R - \frac{6\pi}{3} \int_0^R r^3 \frac{dP}{dr} \, dr$$

Use hydrostatic equilibrium: $\frac{dP}{dr} = -\frac{G M_r \rho}{r^2}$

$$K = 2\pi \int_o^R r^3 \frac{G M_r \rho}{r^2} \, dr = \frac{1}{2} \int \frac{G M_r \rho}{r} 4\pi r^2 \, dr = -\frac{1}{2} U$$

Total Energy: $K + U = \frac{1}{2} U$ *(less than 0)*

What source of energy powers the Sun, and how long will it last?

$L_{\odot} = 4 \times 10^{26}$ W

On the Age of the Sun's Heat

By Sir William Thomson (Lord Kelvin)

Macmillan's Magazine, vol. 5 (March 5, **1862**), pp. 288-293.

How much the sun is actually cooled from year to year... we know from the independent but concordant investigations of Herschel and Pouillet, that the sun radiates every year from his whole surface about 6×10^{30} times as much heat as is sufficient to raise the temperature of one pound of water by 1 °C...

... we may still hold that meteoric action (*heat arising during the formation stage when different parts of the Sun coalesce together*) is not only proved to exist as a cause of solar heat, but it is the only one of all conceivable causes.... That some form of the meteoric theory is certainly the true and complete explanation of solar heat can scarcely be doubted, when the following reasons are considered:

1. No other natural explanation, except by chemical action, can be conceived
2. The chemical theory is insufficient... only generate about 3,000 years' heat.
3. There is no difficulty in accounting for 20 Myrs' heat by the meteoric theory.

It seems, therefore, on the whole most probable that the sun has not illuminated the earth for 100Myrs, and almost certain that he has not done so for 500 Myrs (*Darwin's estimate from the Channel cliff fossils yields 300 Myrs*). As for the future, we may say, with equal certainty, that inhabitants of the earth can not continue to enjoy the light and heat essential to their life for many million years longer unless sources now unknown to us are prepared in the great storehouse of creation.

Exercise: How much energy is in a Mars bar?

$L_\odot$ = 4x10^{26} W, what is its life span?

1) Chemical Energy: assume 1 eV/H-particle (e.g., 2H + O → H_2O)

$\sim M_\odot/m_H$ eV ~ 10^{57} eV ~ 10^{38} J ⇒ 10^4 yrs

2) Gravitational Energy: $\sim -\mathbf{f}\, G M_\odot^2/R_\odot \sim -10^{41}$ J ⇒ 10^7 yrs

3) Thermal Energy: how much is there?

Could the Sun be born hot and just keep cooling?

4) Fission (radioactive element decay): insignificant

earth crust ^{238}Uranium: ~10^{-6} by mass

~1 MeV/nucleon (~1 MeV/A)

5) What else?

Gravito-thermal Collapse, Kelvin-Helmholtz Time

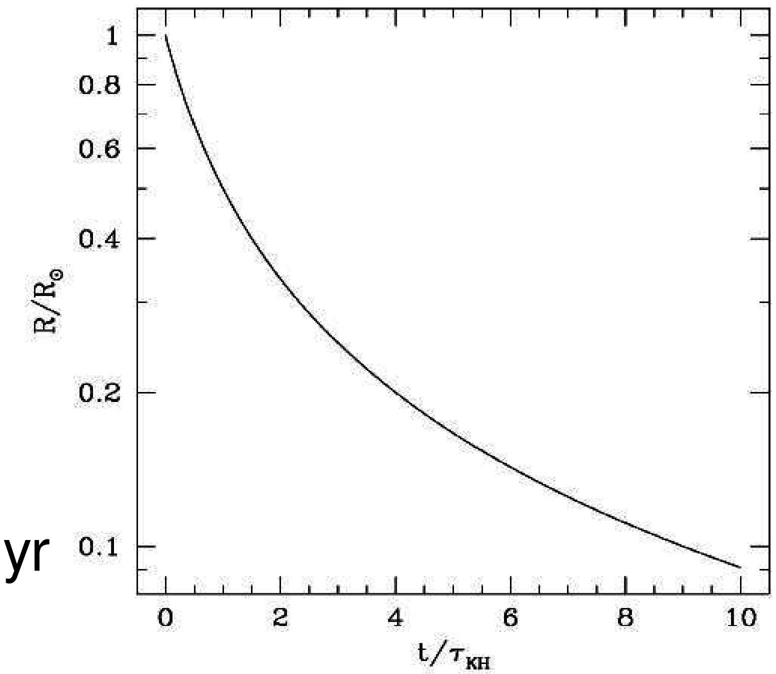
If the Sun has no energy source but has to radiate at constant L

$$\frac{d}{dt}(K+U)=\frac{d}{dt}\left(-\frac{f}{2}\frac{GM^2}{R}\right)=L$$

$$\frac{dR}{dt}=-\frac{L}{fGM^2}R^2 \quad \rightarrow \quad \frac{1}{R}=\frac{L}{fM^2}t+const$$

$$R=R_0 \text{ at } t=0 \quad \rightarrow \quad const=\frac{1}{R_0}$$

$$\frac{R}{R_o}=\frac{1}{1+\frac{t}{\tau_{KH}}}, \quad where \quad \tau_{KH}=\frac{fGM^2/R_0}{L} \sim 10\,\text{Myr}$$



Slowly contracts, releasing gravitational energy (U more negative)

- Contraction proceeds on thermal timescale
  (aka Kelvin-Helmholtz time; *much longer* than dynamical timescale)
- It heats up (KE more positive)
  Gravitational systems have *negative* specific heat
- 1/2 the gravitational energy released goes into KE, the other half....

Virial Theorem

For a bound gravitational system, the long-term average of its kinetic energy is one half that of its potential energy.

Exercise
Use Virial Theorem to estimate central temperature of the Sun
(central material dominates the total kinetic energy)

$$K = \int \frac{3}{2} k_B T \, n \, dV \sim \frac{3}{2} k_B T_c \frac{M}{\mu m_H}$$

$$U = -\int \frac{G M_r}{r} dM_r = -f \frac{GM^2}{R}$$

$$\text{Virial Theorem: } K = -\frac{1}{2} U \quad \rightarrow \quad T_c \sim \frac{f\mu}{3} \frac{GM^2}{R} \frac{m_H}{k_B M} \sim \frac{G M m_H}{k_B R} \sim 10^7 K$$

$$\text{Actual temperature } T_c = 1.6 \times 10^7 K$$

(implications in the next lecture).

Extra Note: proof of Virial Theorem for a rotationally supported system (kinetic energy = orbital or spin energy)

(see Carroll & Ostlie §2.4 for a system of discrete objects.)

$$\textit{Momentum density:}\ \boldsymbol{p}=\rho\boldsymbol{v}=\rho\frac{d\boldsymbol{r}}{dt}$$

$$\textit{Moment of inertia:}\ I=\int\rho r^2\,dV$$

$$\textit{Consider}\ Q=\int\boldsymbol{p}\cdot\boldsymbol{r}\,dV=\frac{1}{2}\frac{d}{dt}\int\rho r^2\,dV=\frac{1}{2}\frac{dI}{dt}$$

$$\frac{dQ}{dt}=\int\frac{d\boldsymbol{p}}{dt}\cdot\boldsymbol{r}\,dV+\int\boldsymbol{p}\cdot\boldsymbol{v}\,dV$$

$$\int\frac{d\boldsymbol{p}}{dt}\cdot\boldsymbol{r}\,dV=\frac{\int\boldsymbol{F}}{\boldsymbol{r}}dV=\int\int dV_1\,dV_2\frac{G\rho_1\rho_2}{|\boldsymbol{r}_2-\boldsymbol{r}_1|}=\int\frac{GM_r\rho}{r}dV=U$$

$$\int\boldsymbol{p}\cdot\boldsymbol{v}\,dV=2K$$

$$\textit{So,}\ 2\langle K\rangle+\langle U\rangle=\langle\frac{d^2I}{dt^2}\rangle=0$$

$$\textit{In particular, for a Keplerian orbit:}\ \langle E_{tot}\rangle=\langle\frac{1}{2}mv^2\rangle-\langle\frac{GMm}{r^2}\rangle=-\frac{GMm}{2a}$$