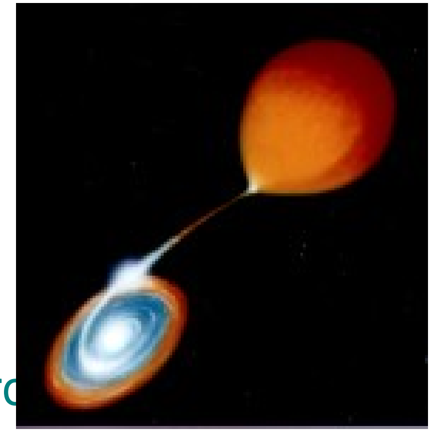
Laws of Gravity II

Kepler's three laws applicable if m<<M *(one moving body, 1-body)*

What if m & M are comparable?

Two moving bodies (2-body)
Use '***reduced mass***'



applications: binary stars, galaxies, (dwarf) planets (pluto-charc
detecting extra-solar planets, ...

Three-body...

N-body...

General problem

$$\frac{d\mathbf{r}_i}{dt}=\mathbf{v}_i, \quad \frac{d\mathbf{v}_i}{dt}=a_i=\sum_{j\neq i}\frac{\mathbf{F}_{ji}}{m_i} \quad \text{Hence,} \quad \frac{d^2\mathbf{r}_i}{dt^2}=\sum_{j\neq i}\frac{Gm_j}{|r_j-r_i|^2}\frac{\mathbf{r}_j-\mathbf{r}_i}{|r_j-r_i|}$$

Nx3 coupled second-order differential equations!

1-body: *For* $N=2$ *with* $m_1 \gg m_2$*, we approximate* $\mathbf{r}_1 \simeq \mathbf{0}$

$$\text{So that } \frac{d^2\mathbf{r}}{dt^2}=-\frac{GM}{r^2}\frac{\mathbf{r}}{r}$$

with $\mathbf{r}=\mathbf{r}_2$, $M=m_1$*, and* $m=m_2$

Solving this yields:

$$L=m\sqrt{GMa(1-e^2)}$$

$$E=-\frac{GMm}{2a}$$

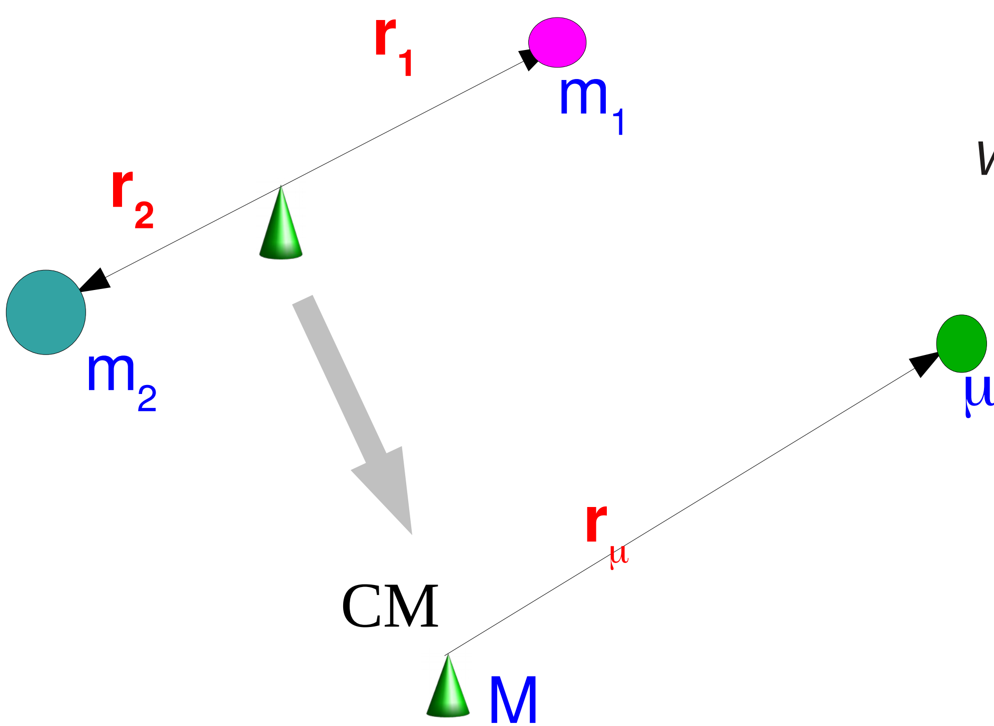
$$\frac{dA}{dt}=\frac{L}{2m}=const$$

$$P^2=\frac{4\pi^2}{GM}a^3 \quad (K\ III)$$

Two-body problem:

reduce it to an equivalent one-body problem using the concept of '***reduced mass***':

An imaginary particle of mass μ, distance $\boldsymbol{r}$ away from CM *(center of mass)* 'the reduced mass particle'



$$Let\ \boldsymbol{r}_\mu = \boldsymbol{r}_1 - \boldsymbol{r}_2,\quad M = m_1 + m_2$$

$$Let\ \boldsymbol{r}_\mu, \boldsymbol{r}_1, \boldsymbol{r}_2\ be\ measured\ from\ CM$$

$$with\ m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2 = 0$$

$$We\ have\ \boldsymbol{r}_1 = \frac{m_2}{M}\boldsymbol{r}_\mu,\quad \boldsymbol{r}_2 = -\frac{m_1}{M}\boldsymbol{r}_\mu,\ and$$

$$|\boldsymbol{F}_{12}| = |\boldsymbol{F}_{21}| = \frac{G m_1 m_2}{|r_1 - r_2|^2} = \frac{G \mu M}{r_\mu^2}$$

$$where\ \mu = \frac{m_1 m_2}{m_1 + m_2}$$

2-body: Energy, Angular Momentum & Kepler's Laws

$$\textit{Angular momentum } \mathbf{L} = m_1 \mathbf{r}_1 \times \mathbf{v}_1 + m_2 \mathbf{r}_2 \times \mathbf{v}_2 = \mu \mathbf{r}_\mu \times \mathbf{v}_\mu$$

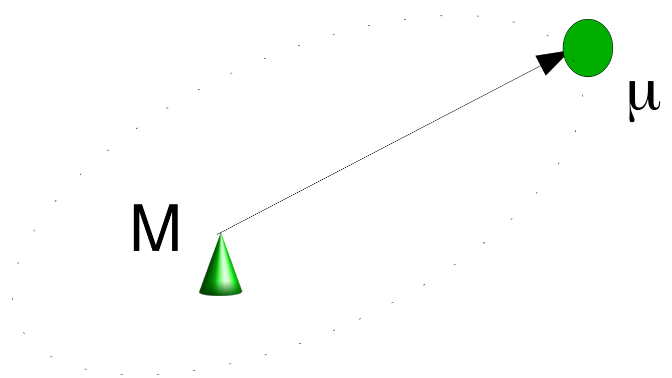
$$\left(\textit{where } \mathbf{v}_\mu = \frac{d\mathbf{r}_\mu}{dt} = \mathbf{v}_1 - \mathbf{v}_2\right)$$

$$\textit{Energy } E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{G m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{1}{2} \mu v_\mu^2 - \frac{G \mu M}{r_\mu}$$

Hence, as for 1-body case $(m \ll M)$, *we have*

$$L = \mu \sqrt{G M a_\mu (1 - e^2)}, \quad E = -\frac{G \mu M}{2 a_\mu}$$

together with Kepler's three Laws, e.g., $P^2 = \dfrac{4\pi^2}{GM} a_\mu^3$



General problem

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Nx3 coupled second-order differential equations!

1-body:

For $N=2$ with $m_1 \gg m_2$, we approximate $\mathbf{r}_1 \simeq \mathbf{0}$

$$\text{So that } \frac{d^2\mathbf{r}}{dt^2}=-\frac{GM}{r^2}\frac{\mathbf{r}}{r}$$

with $\mathbf{r}\simeq\mathbf{r}_2$, $M\simeq m_1$, and $\mu\simeq m_2$

2-body:

For $N=2$ with $m_1 \sim m_2$, we use $m_1\mathbf{r}_1+m_2\mathbf{r}_2\equiv\mathbf{0}$

$$\text{So that } \frac{d^2\mathbf{r}}{dt^2}=-\frac{GM}{r^2}\frac{\mathbf{r}}{r}$$

with $\mathbf{r}\equiv\mathbf{r}_2-\mathbf{r}_1$, $M\equiv m_1+m_2$, and $\mu\equiv\frac{m_1 m_2}{m_1+m_2}$

$$L=\mu\sqrt{GMa(1-e^2)}$$

$$E=-\frac{GM\mu}{2a}$$

$$\frac{dA}{dt}=\frac{L}{2\mu}=const$$

$$P^2=\frac{4\pi^2}{GM}a^3 \quad (K\ III)$$

Kepler in the astronomer's toolkit

$$GM = a^3 \left(\frac{2\pi}{P} \right)^2 = a^3 \Omega^2$$

$$v_{tot,circ} = \sqrt{\frac{GM}{a}}$$

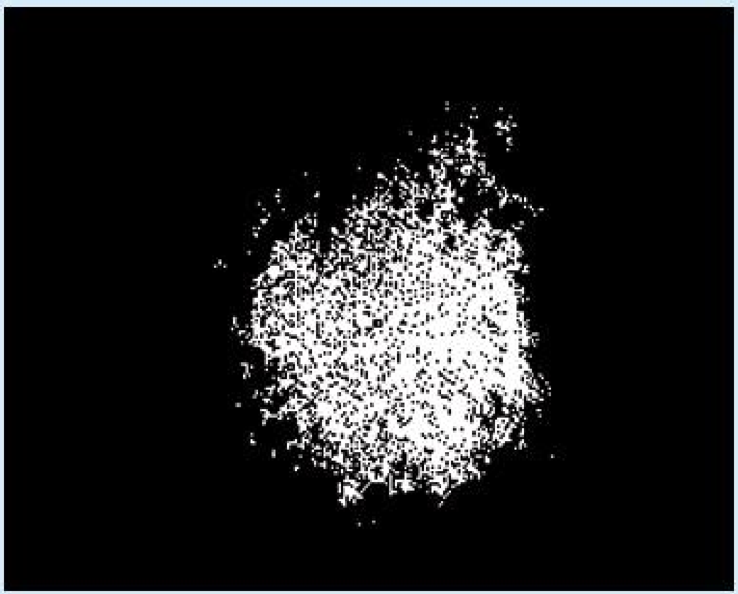
$$\frac{a_1}{a_2} = \frac{v_1}{v_2} = \frac{m_2}{m_1}, \quad \frac{a_1}{a} = \frac{v_1}{v} = \frac{m_2}{M}$$

Determine masses in binary systems

$M_{pluto} + M_{charon} \sim 1/400\ M_E$

$M_{Charon}/M_{Pluto} \sim 1/7$

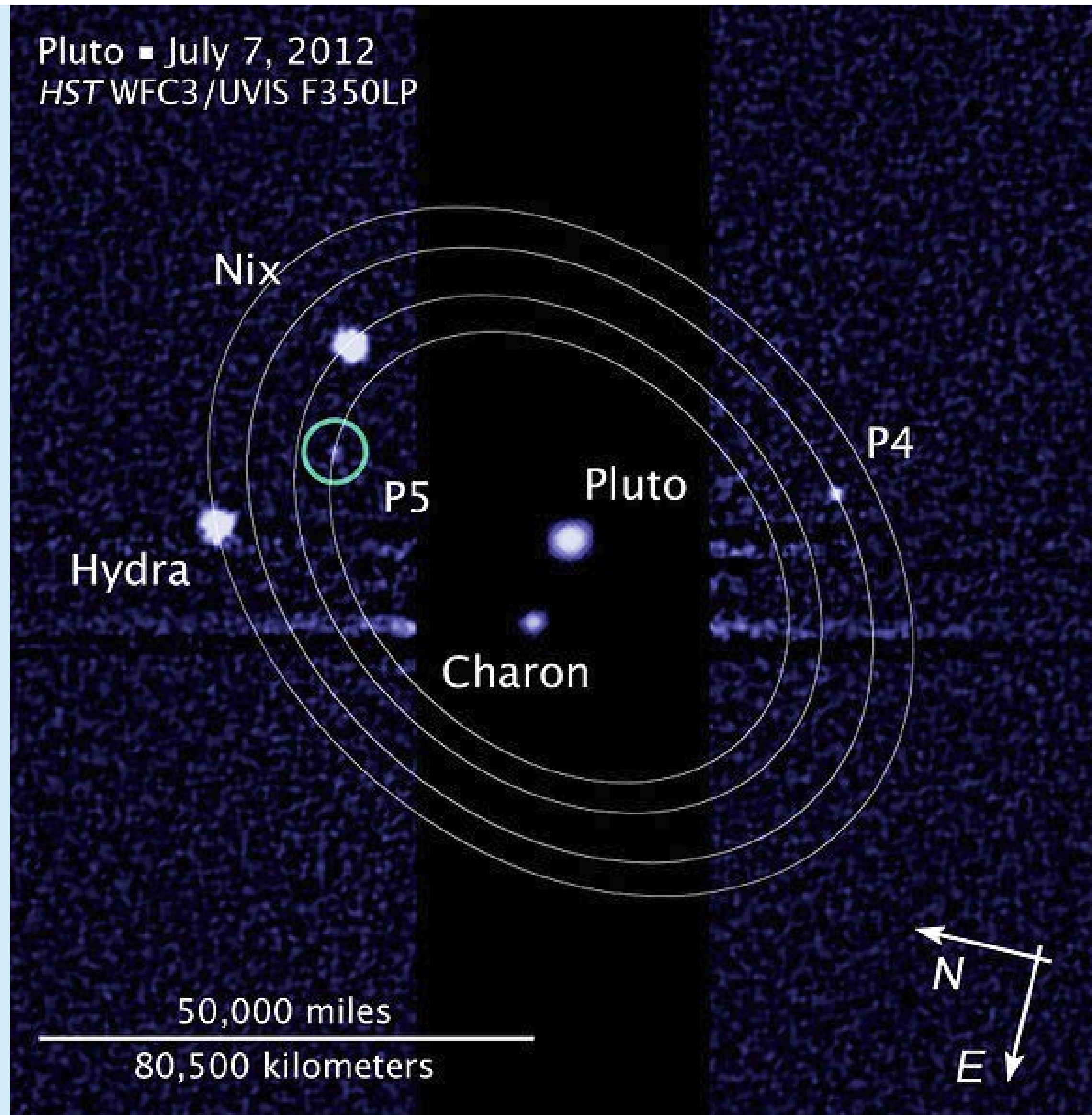
BREAKTHROUGH 1978: CHARON
(Christy and Harrington)



Pluto's Satellite Charon

Orbital Parameters:

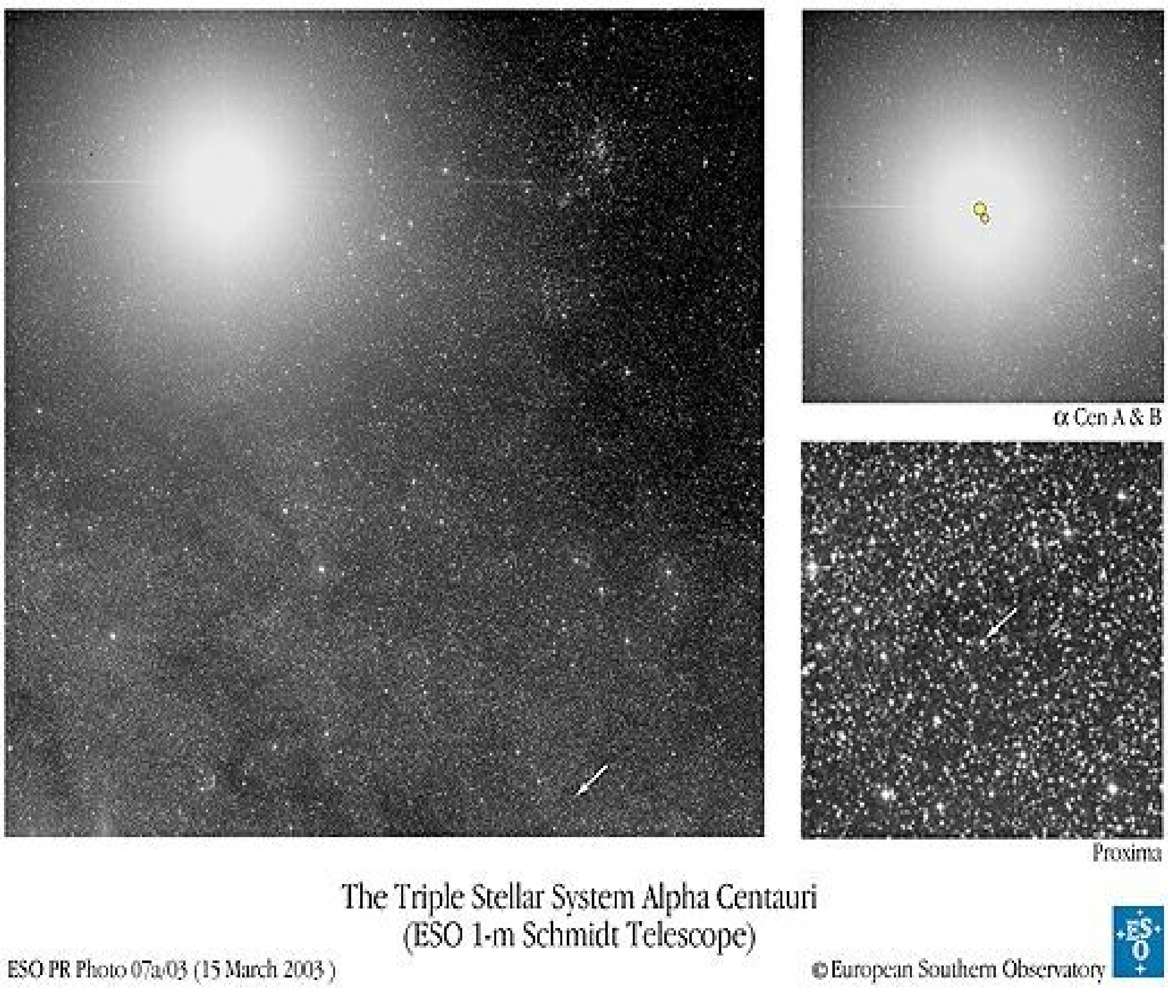
- Distance from Pluto = 17,600 km
- Orbit Period = 6.3867 days
- Circular orbit



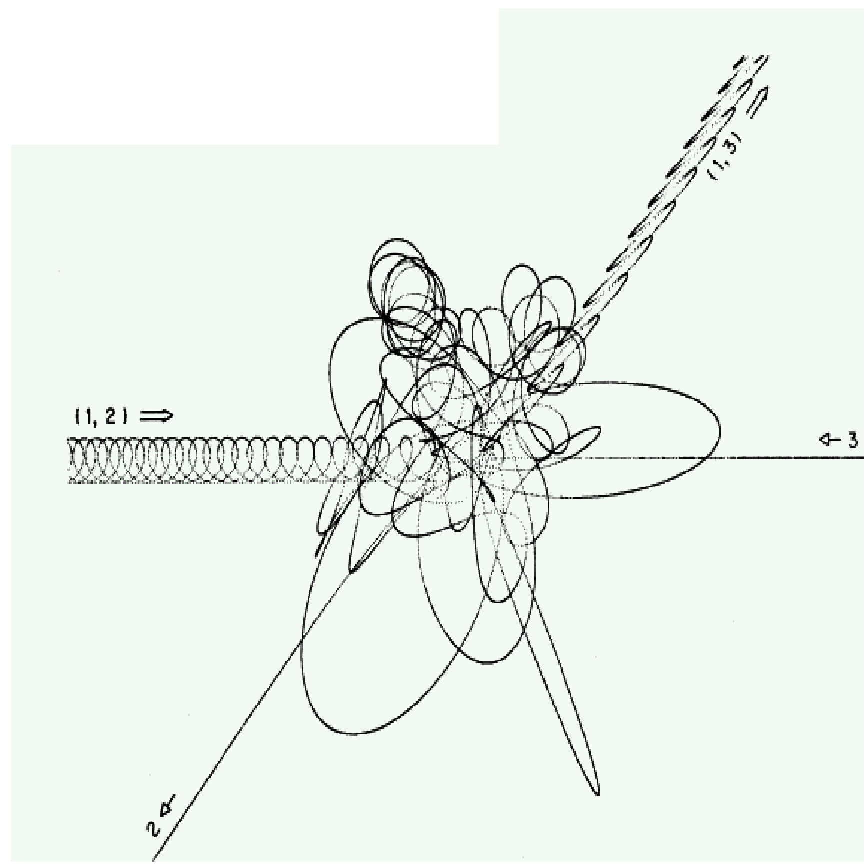
Three-body gravitational interaction:

not reducible to one-body
In general, motion no longer periodic (quasi-periodic or chaotic)

our closest neighbour α-Centauri (a triple system)



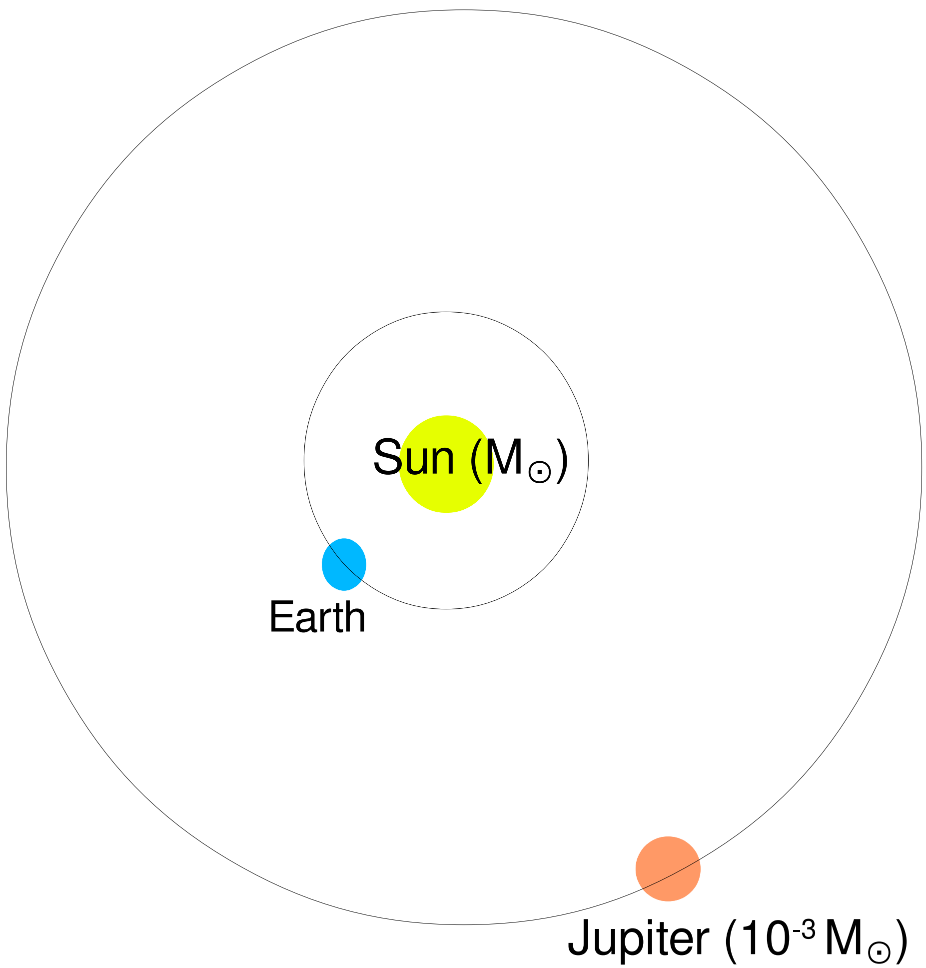
triple star encounter simulation
(Hut)

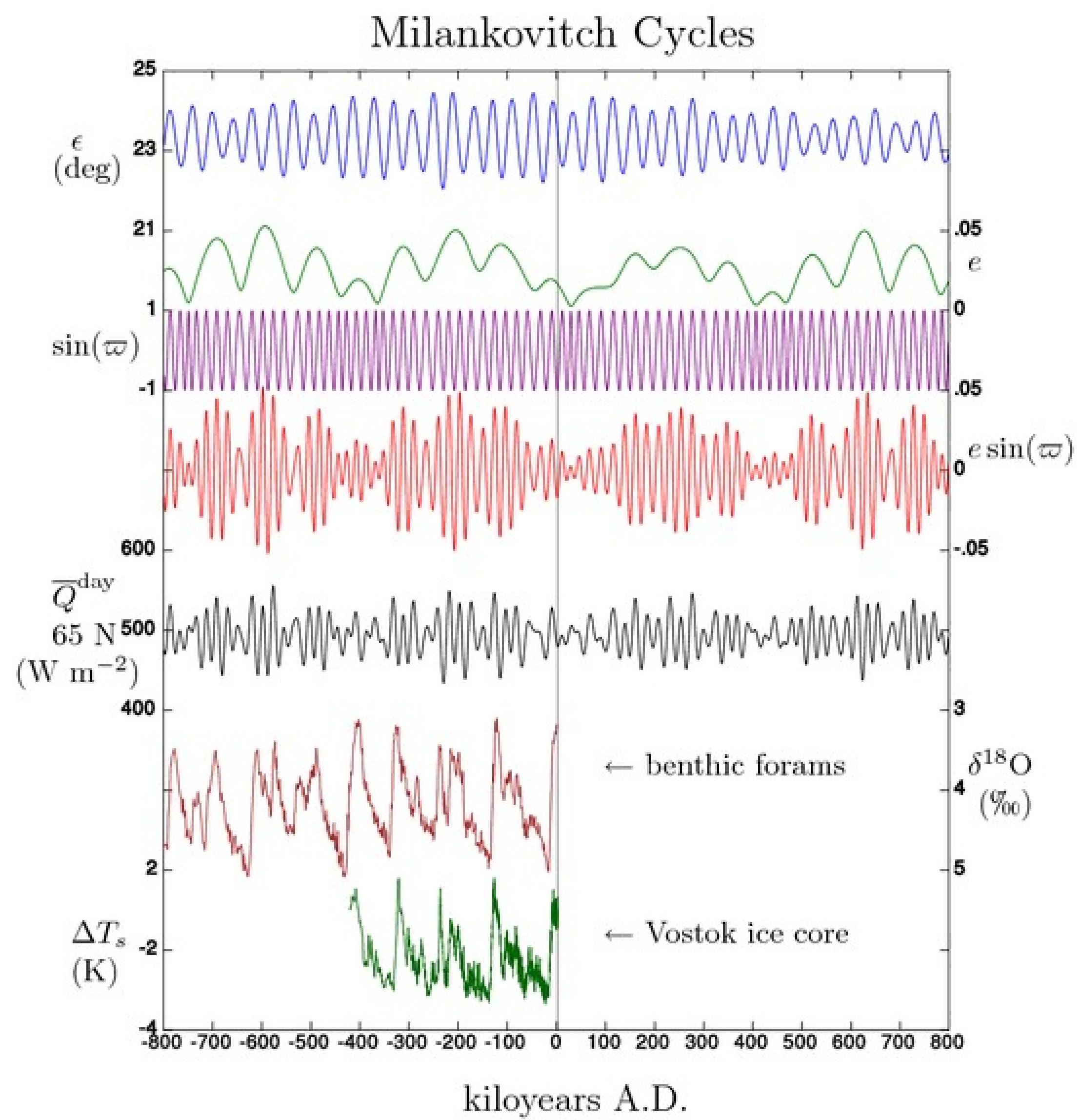


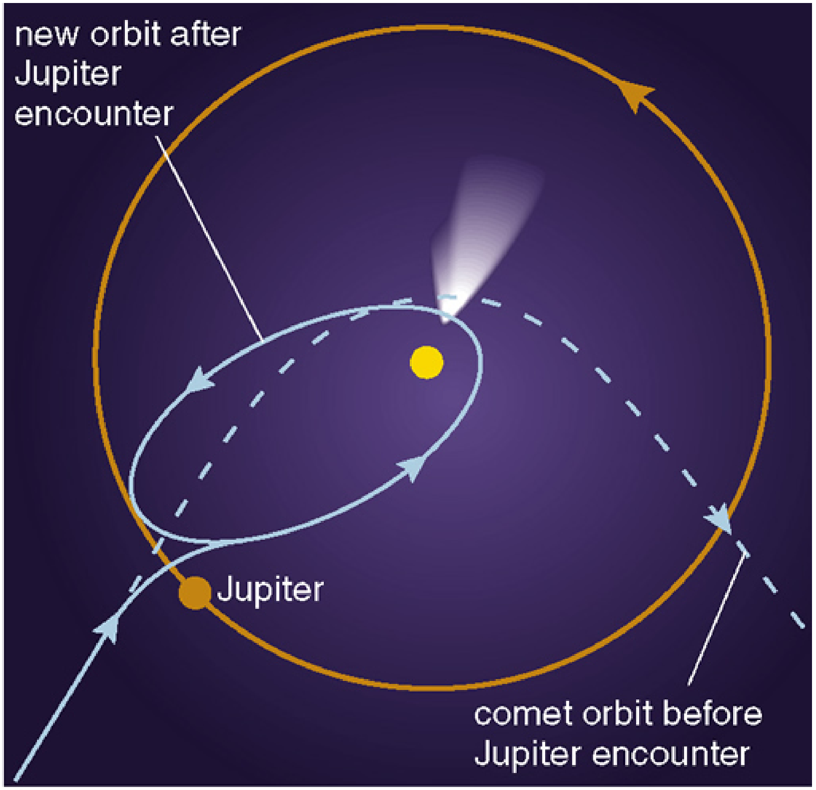
3-body (practical applications):

- Stability of the Solar system

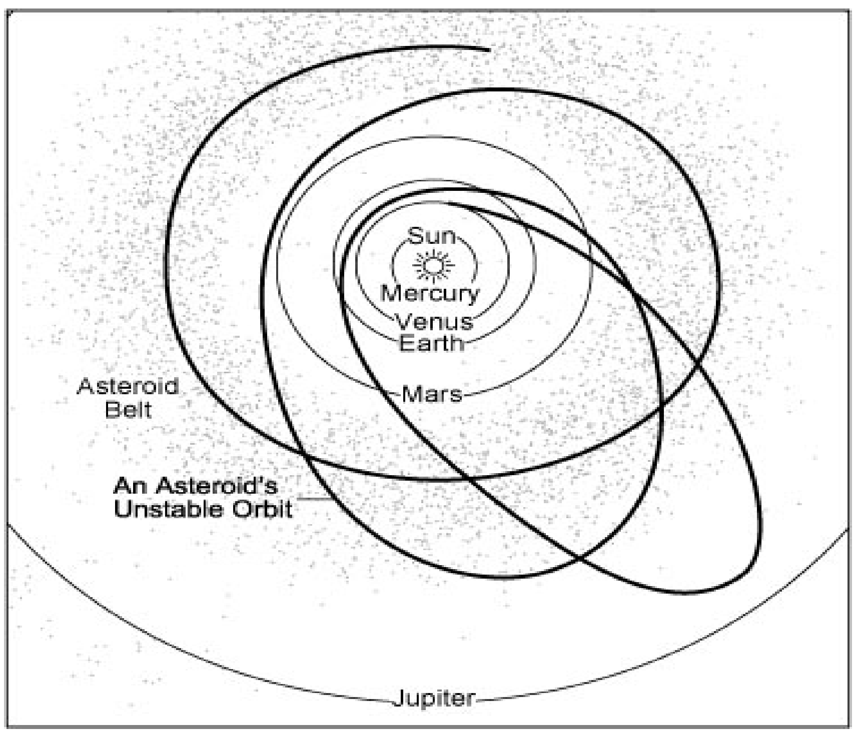
  very rich phenomenology & analytical theory -- Planetary Dynamics
- Climate changes on Earth



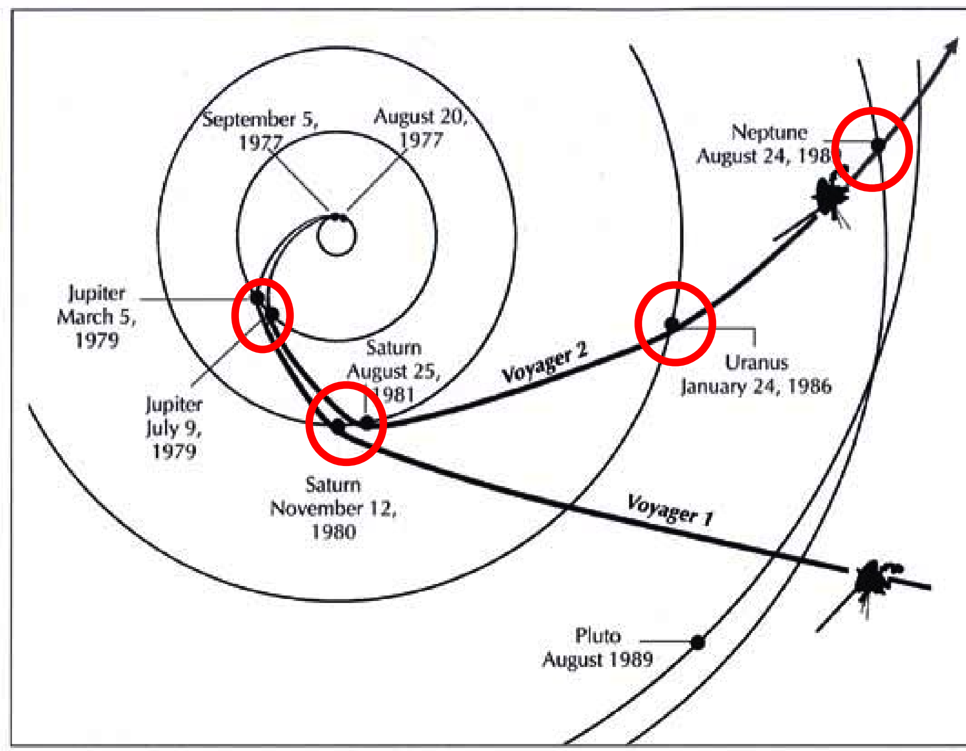




Orbital evolution of near-earth asteroids and comets.



Space travel

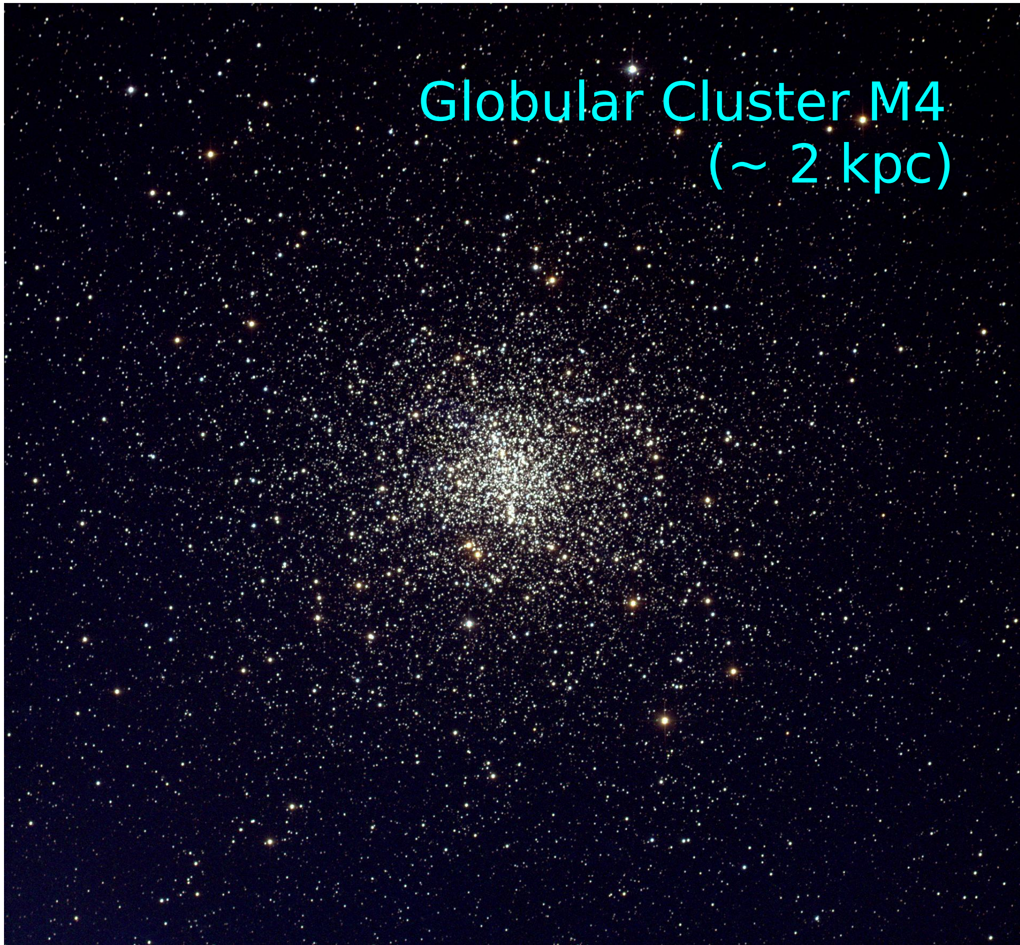


Voyagers getting swings from outer planets -- gravity-assist

N-body

intractable analytically
relies on numerical integration

globular cluster	~ 10^6 stars
cluster of galaxies	~ 10^3 galaxies
Universe	~ 10^{11} galaxies



Quiz: cannibalism in close binary stars

star m_1 bloats up, part (d m) of its envelope becomes dominated by the gravity of m_2 and is transferred from m_1 to m_2

-- does the binary unbind or spiral-in?

How to estimate?

- Use energy conservation?
- Use angular momentum conservation?

Cataclysmic Variable