Laws of Gravity II

Kepler's three laws applicable if m<<M *(one moving body, 1-body)*

What if m & M are comparable?

Two moving bodies (2-body)
Use '***reduced mass***'

applications: binary stars, galaxies, (dwarf) planets (pluto-charon),
detecting extra-solar planets, ...

Three-body...

N-body...

General problem

$$\frac{d\mathbf{r}_i}{dt}=\mathbf{v}_i, \quad \frac{d\mathbf{v}_i}{dt}=a_i=\sum_{j\neq i}\frac{\mathbf{F}_{ji}}{m_i} \quad \text{Hence,} \quad \frac{d^2\mathbf{r}_i}{dt^2}=\sum_{j\neq i}\frac{G m_j}{|r_j-r_i|^2}\frac{\mathbf{r}_j-\mathbf{r}_i}{|r_j-r_i|}$$

Nx3 coupled second-order differential equations!

1-body: *For* $N=2$ *with* $m_1 \gg m_2$*, we approximate* $\mathbf{r}_1 \simeq \mathbf{0}$

$$\text{So that} \quad \frac{d^2\mathbf{r}}{dt^2}=-\frac{GM}{r^2}\frac{\mathbf{r}}{r}$$

with $\mathbf{r}=\mathbf{r}_2$, $M=m_1$*, and* $m=m_2$

Solving this yields:

$$L=m\sqrt{GMa(1-e^2)}$$

$$E=-\frac{GMm}{2a}$$

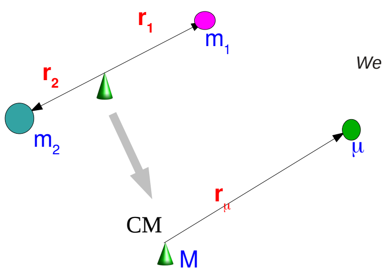
$$\frac{dA}{dt}=\frac{L}{2m}=const$$

$$P^2=\frac{4\pi^2}{GM}a^3 \quad (K\ III)$$

Two-body problem:

reduce it to an equivalent one-body problem
using the concept of '***reduced mass***':

An imaginary particle of mass μ,
distance $\boldsymbol{r}$ away from CM *(center of mass)*
'the reduced mass particle'



Let $\mathbf{r}_\mu=\mathbf{r}_1-\mathbf{r}_2$, $M=m_1+m_2$

Let $\mathbf{r}_\mu, \mathbf{r}_1, \mathbf{r}_2$ *be measured from CM*

with $m_1\mathbf{r}_1+m_2\mathbf{r}_2=0$

We have $\mathbf{r}_1=\frac{m_2}{M}\mathbf{r}_\mu$, $\mathbf{r}_2=-\frac{m_1}{M}\mathbf{r}_\mu$*, and*

$$|\mathbf{F}_{12}|=|\mathbf{F}_{21}|=\frac{G m_1 m_2}{|r_1-r_2|^2}=\frac{G\mu M}{r_\mu^2}$$

where $\mu=\frac{m_1 m_2}{m_1+m_2}$

2-body: Energy, Angular Momentum & Kepler's Laws

Angular momentum $\mathbf{L}=m_1\mathbf{r}_1\times\mathbf{v}_1+m_2\mathbf{r}_2\times\mathbf{v}_2=\mu\,\mathbf{r}_\mu\times\mathbf{v}_\mu$

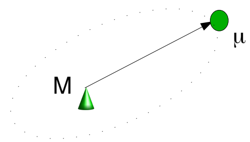
(where $\mathbf{v}_\mu=\frac{d\mathbf{r}_\mu}{dt}=\mathbf{v}_1-\mathbf{v}_2$*)*

$$\text{Energy } E = \frac{1}{2}m_1 v_1^2+\frac{1}{2}m_2 v_2^2-\frac{G m_1 m_2}{|\mathbf{r}_1-\mathbf{r}_2|}=\frac{1}{2}\mu v_\mu^2-\frac{G\mu M}{r_\mu}$$

Hence, as for 1-body case $(m \ll M)$*, we have*

$$L=\mu\sqrt{GMa_\mu(1-e^2)},\ E=-\frac{G\mu M}{2a_\mu}$$

together with Kepler's three Laws, e.g., $P^2=\frac{4\pi^2}{GM}a_\mu^3$



General problem

$\frac{d\mathbf{r}_i}{dt}=\mathbf{v}_i, \quad \frac{d\mathbf{v}_i}{dt}=a_i=\sum_{j\neq i}\frac{F_{ji}}{m_i}$ *Hence,* $\frac{d^2\mathbf{r}_i}{dt^2}=\sum_{j\neq i}\frac{Gm_j}{|r_j-r_i|^2}\frac{\mathbf{r}_j-\mathbf{r}_i}{|r_j-r_i|}$

Nx3 coupled second-order differential equations!

1-body:

For $N=2$ *with* $m_1 \gg m_2$, *we approximate* $\mathbf{r}_1 \simeq \mathbf{0}$

So that $\frac{d^2\mathbf{r}}{dt^2}=-\frac{GM}{r^2}\frac{\mathbf{r}}{r}$

with $\mathbf{r}\simeq\mathbf{r}_2$, $M\simeq m_1$, *and* $\mu\simeq m_2$

2-body:

For $N=2$ *with* $m_1 \sim m_2$, *we use* $m_1\mathbf{r}_1+m_2\mathbf{r}_2\equiv\mathbf{0}$

So that $\frac{d^2\mathbf{r}}{dt^2}=-\frac{GM}{r^2}\frac{\mathbf{r}}{r}$

with $\mathbf{r}\equiv\mathbf{r}_2-\mathbf{r}_1$, $M\equiv m_1+m_2$, *and* $\mu\equiv\frac{m_1 m_2}{m_1+m_2}$

$$L=\mu\sqrt{GMa(1-e^2)}$$

$$E=-\frac{GM\mu}{2a}$$

$$\frac{dA}{dt}=\frac{L}{2\mu}=const$$

$$P^2=\frac{4\pi^2}{GM}a^3 \quad \text{(K III)}$$

Kepler in the astronomer's toolkit

$$GM=a^3\left(\frac{2\pi}{P}\right)^2=a^3\Omega^2$$

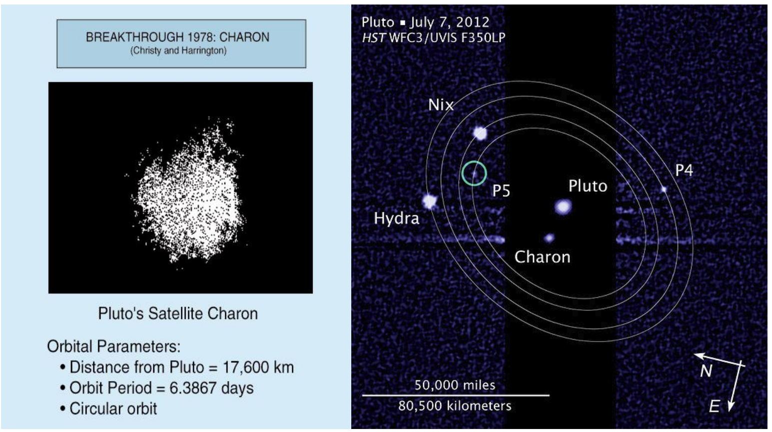
$$v_{tot,circ}=\sqrt{\frac{GM}{a}}$$

$$\frac{a_1}{a_2}=\frac{v_1}{v_2}=\frac{m_2}{m_1}, \quad \frac{a_1}{a}=\frac{v_1}{v}=\frac{m_2}{M}$$

Determine masses in binary systems

$M_{pluto} + M_{charon} \sim 1/400\ M_E$

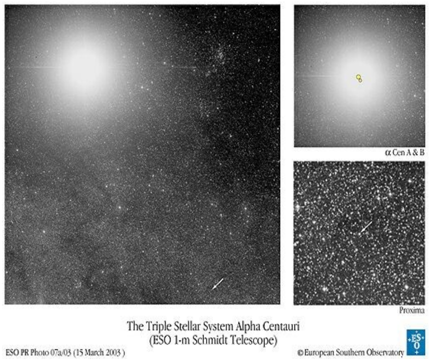
$M_{Charon}/M_{Pluto} \sim 1/7$



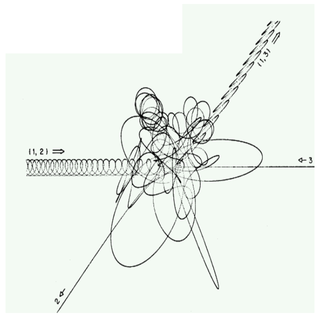
Three-body gravitational interaction:

not reducible to one-body
In general, motion no longer periodic (quasi-periodic or chaotic)

our closest neighbour α-Centauri (a triple system)

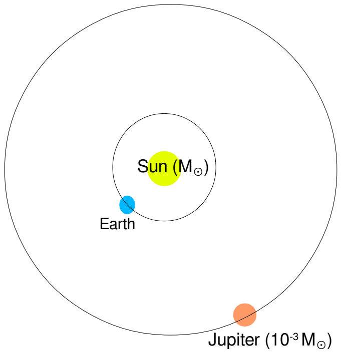


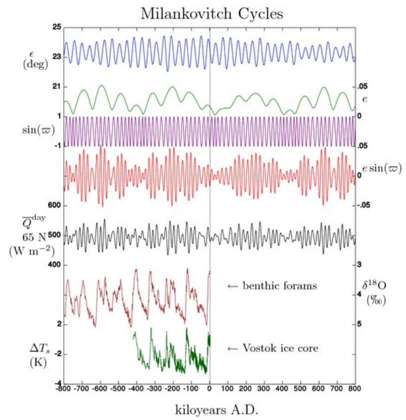
triple star encounter simulation (Hut)

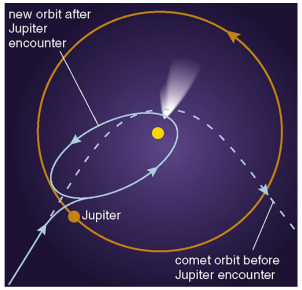


3-body (practical applications):

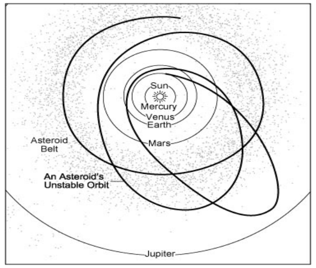
- Stability of the Solar system
  very rich phenomenology & analytical theory -- Planetary Dynamics
- Climate changes on Earth



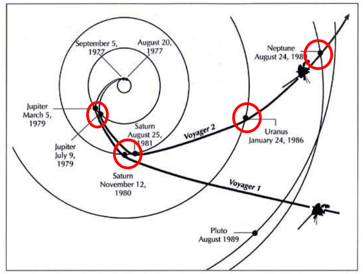




Orbital evolution of near-earth asteroids and comets.



Space travel



Voyagers getting swings from outer planets -- gravity-assist

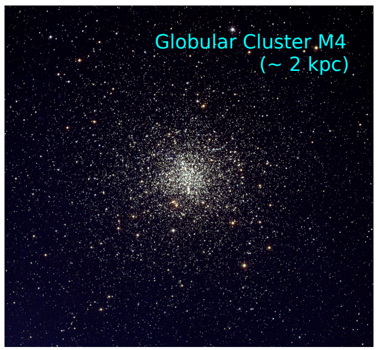
N-body

intractable analytically
relies on numerical integration

globular cluster ~ 10^6 stars

cluster of galaxies ~ 10^3 galaxies

Universe ~ 10^{11} galaxies



Quiz: cannibalism in close binary stars

star m_1 bloats up, part (d m) of its envelope becomes dominated by the gravity of m_2 and is transferred from m_1 to m_2
-- does the binary unbind or spiral-in?

How to estimate?

- Use energy conservation?
- Use angular momentum conservation?

Cataclysmic Variable