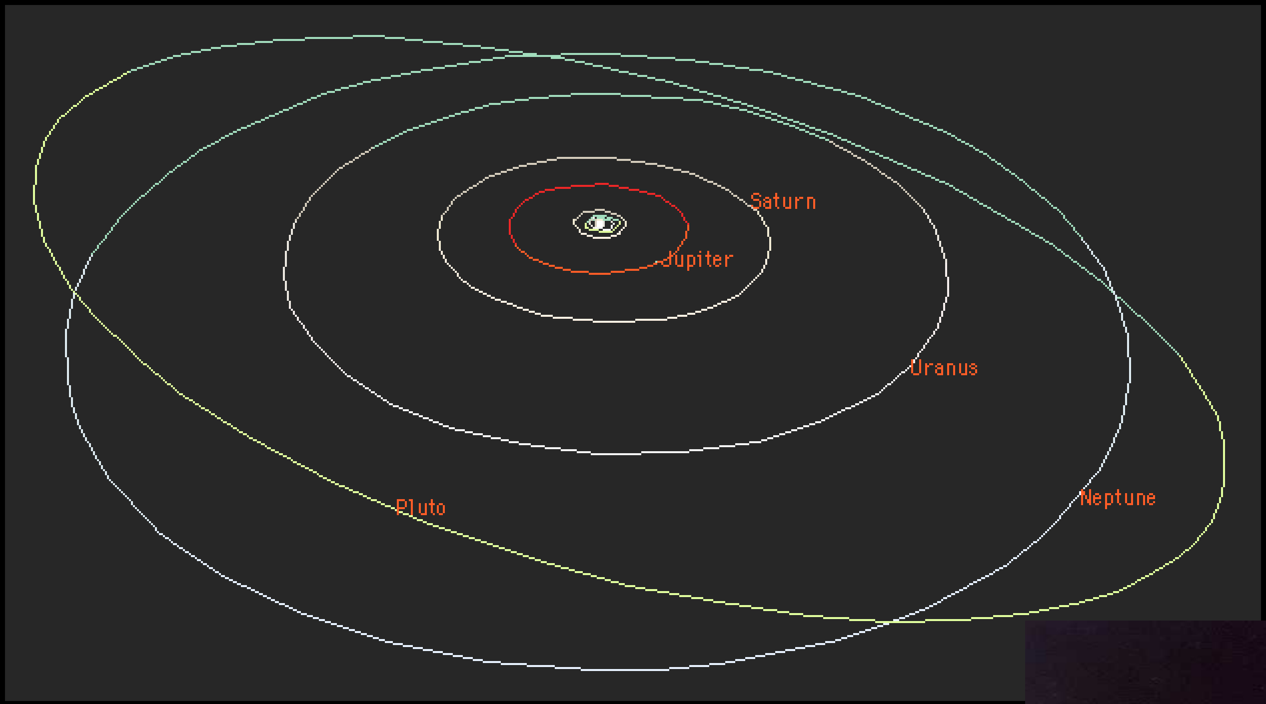
Laws of Gravity I

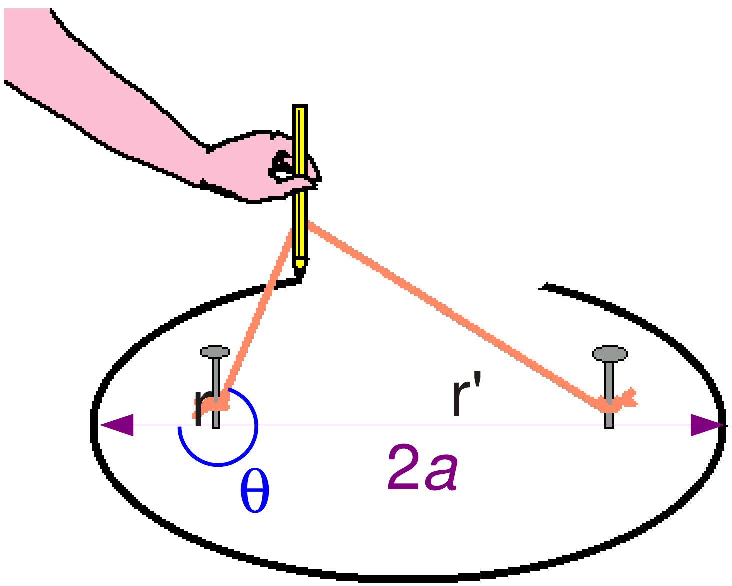


Kepler's (1571-1630) three laws (rules) for planetary motion

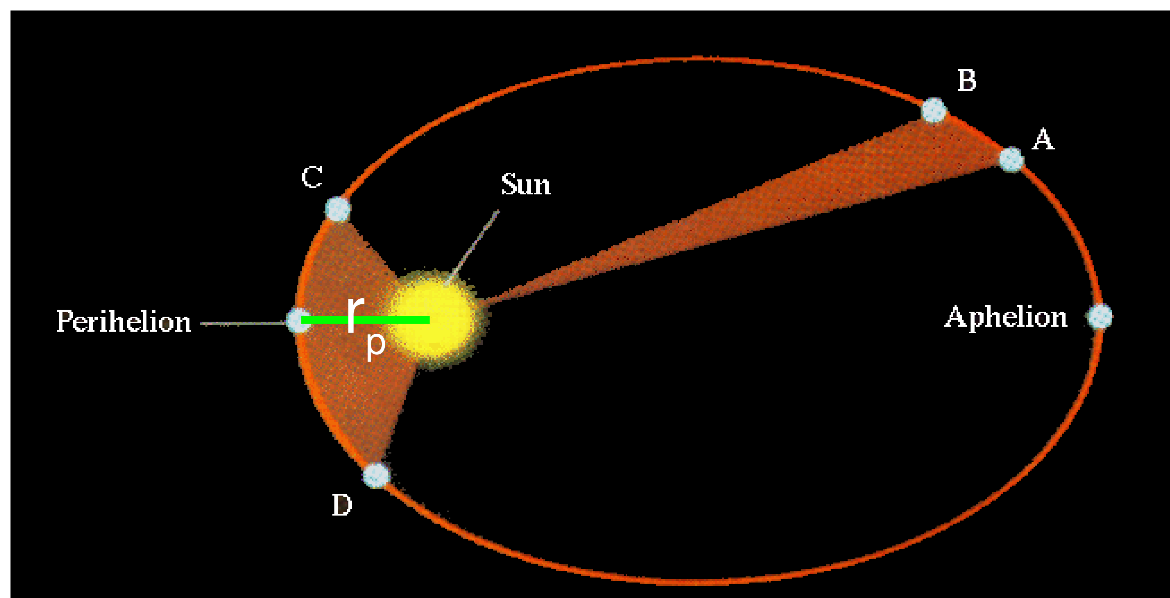
I. A planet orbits the sun in an ellipse with the sun at one focus

II. Planet-sun line sweeps out equal area in equal time intervals

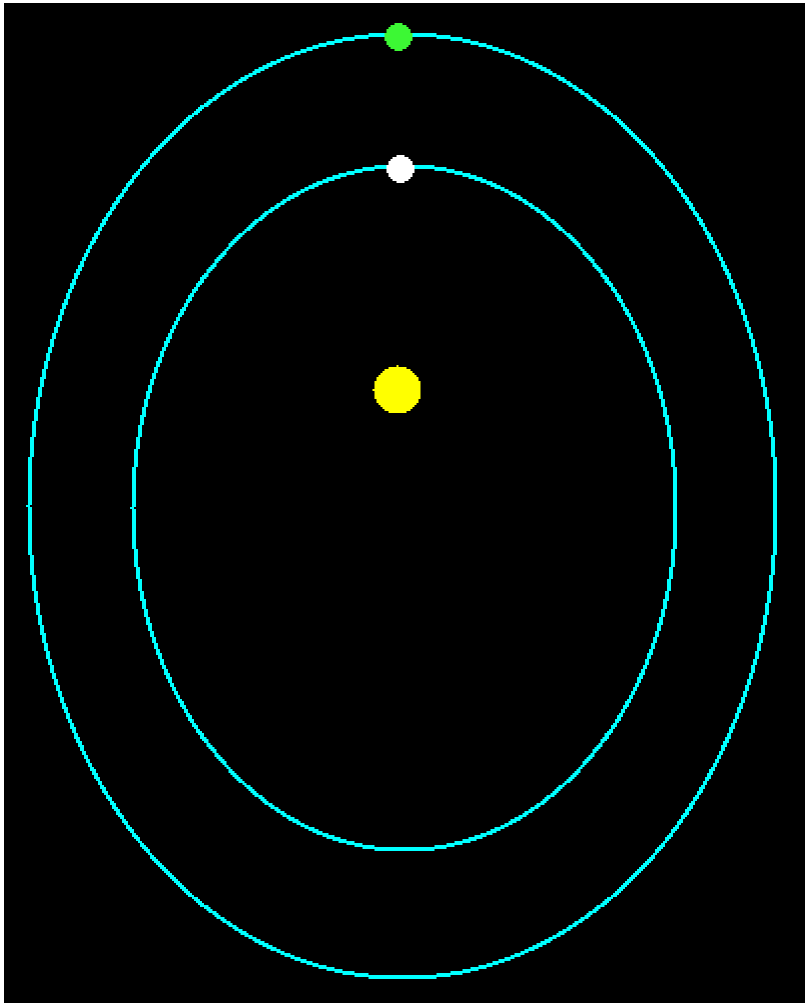
III. $P^2 = a^3$ (for the Sun, with P in years, a in AU)



$$r_p = a(1-e)$$
$$r_a = a(1+e)$$
$$A = \pi a^2 \sqrt{1-e^2}$$



Kepler III: $P^2 = a^3$

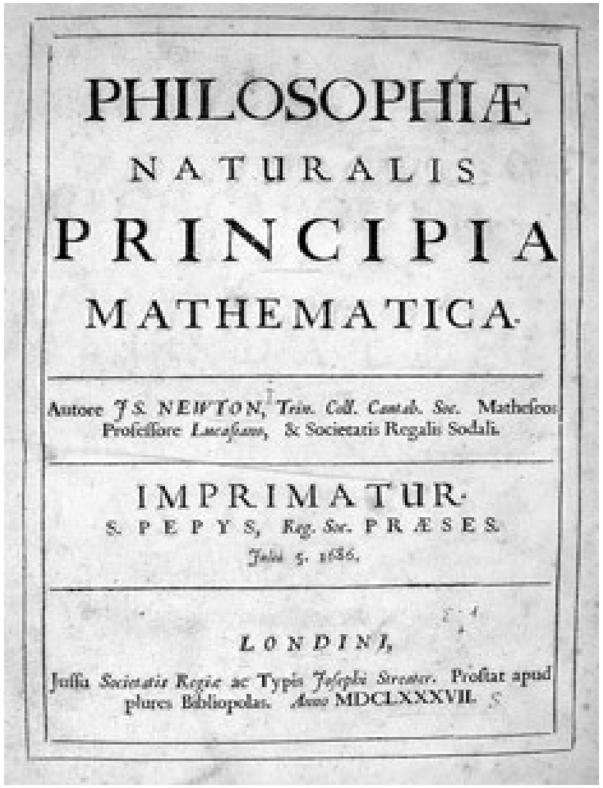


Newton's (1642-1727) three great **laws**

I. the law of inertia

II. $\boldsymbol{F} = m\,\boldsymbol{a}$

III. $\boldsymbol{F}_{12} = -\,\boldsymbol{F}_{21}$



PHILOSOPHIÆ
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MATHEMATICA.

Autore J S. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, Reg. Soc. PRÆSES.
Julii 5. 1686.

LONDINI,
Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud plures Bibliopolas. Anno MDCLXXXVII.

Newton's law of universal gravitation *(tested broadly)*

$$F = G\frac{M\,m}{r^2}$$

Now: onto deriving Kepler's laws...

Gravity only, hence conservation of

- momentum P,
- angular momentum L,
- energy E.

Linear momentum $\boldsymbol{P}=\sum_k \boldsymbol{P}_k=\sum_k m_k \boldsymbol{v}_k$

$$\frac{d\boldsymbol{P}_k}{dt}=\sum_i \boldsymbol{F}_{ik}, \quad \boldsymbol{F}_{ik}=-\boldsymbol{F}_{ki}$$

hence, $\boxed{\frac{d\boldsymbol{P}}{dt}=\frac{d\sum_k \boldsymbol{P}_k}{dt}=0}$

Angular momentum $\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{P}$

$$\frac{d\boldsymbol{L}}{dt}=\frac{d\boldsymbol{r}}{dt}\times\boldsymbol{P}+\boldsymbol{r}\times\frac{d\boldsymbol{P}}{dt}=\boldsymbol{v}\times\boldsymbol{P}+\boldsymbol{r}\times\boldsymbol{F}$$

for gravity $\boldsymbol{F}=-G\frac{Mm}{r^2}\frac{\boldsymbol{r}}{r}$

hence, $\boxed{\frac{d\boldsymbol{L}}{dt}=0}$

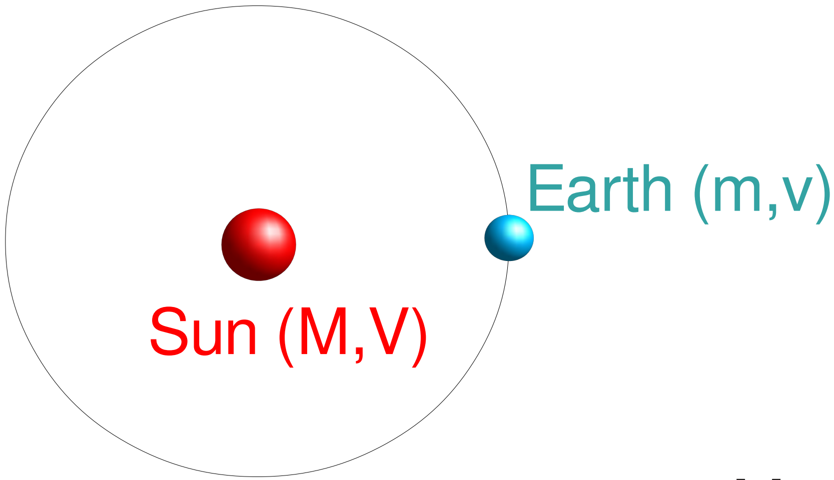
Total energy $E=E_{kin}+E_{pot}$

$$E_{kin}=\frac{1}{2}mv^2, E_{pot}=m\Phi=-m\frac{GM}{r}$$

$$\frac{dE_{kin}}{dt}=\boldsymbol{F}\cdot\boldsymbol{v}=-m\nabla\Phi\cdot\frac{d\boldsymbol{r}}{dt}=-m\frac{d\Phi}{dt}=-\frac{dE_{pot}}{dt}$$

hence, $\boxed{\frac{dE}{dt}=\frac{dE_{kin}+dE_{pot}}{dt}=0}$

Deriving Kepler's Laws using conservation of P, L & E



$m << M$ *(see next lecture for $m \sim M$)*

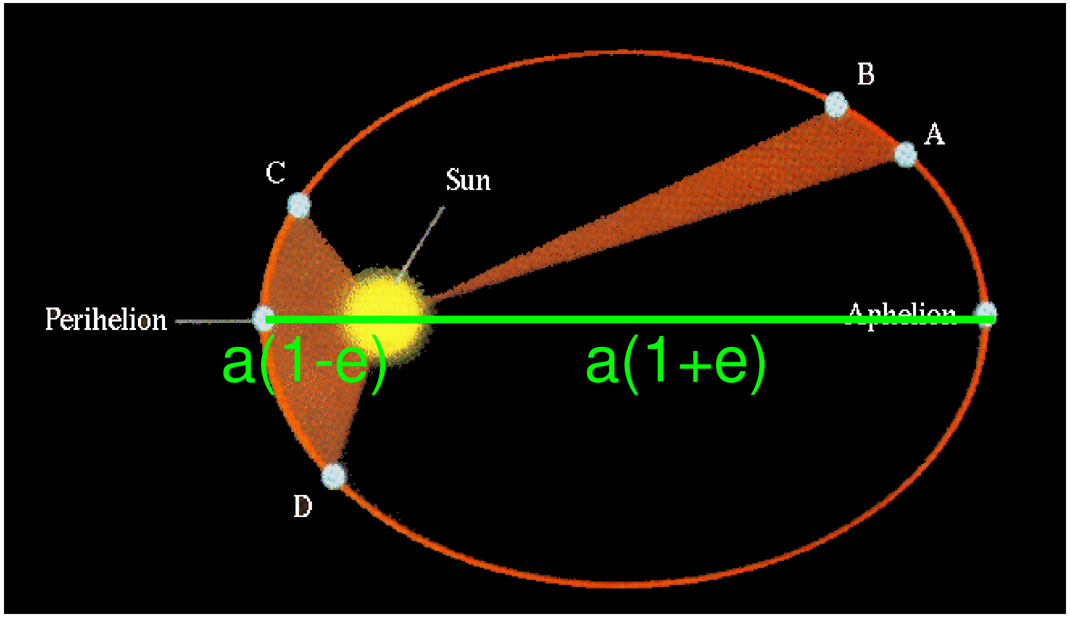
$\mathbf{P} = m\,\mathbf{v} + M\,\mathbf{V}$ = constant *(= 0 if relative to center of mass)*

⇒ reflex motion of the Sun very small
So, we will ignore **V** for L and E
(but not for detecting extra-solar planets!)

$\mathbf{L} = \mathbf{r} \times \mathbf{P} = \mathbf{r} \times m\,\mathbf{v}$ = constant

⇒ Motion in a plane (K I partial) *(for remainder, see book)*

⇒ constant swept-out area per unit time (K II)



Conservation of P, L, E

$\Rightarrow$ Kepler's 3rd law

at perihelion, aphelion,

$$L = m\,r\,v = constant$$

$$E = \frac{1}{2} m v^2 - \frac{G M m}{r} = constant$$

since $r_p = a(1-e)$, $r_a = a(1+e)$

$$v_p^2 = \frac{GM}{a}\left(\frac{1+e}{1-e}\right) \qquad v_a^2 = \frac{GM}{a}\left(\frac{1-e}{1+e}\right)$$

also, $\boxed{L = m\sqrt{G M a (1-e^2)}, \quad E = -\frac{G M m}{2a}}$

$$\frac{dA}{dt} = \frac{\mathbf{L}}{2\mathrm{m}} = constant = \frac{A}{P}$$

for an ellipse, $A = \pi a^2 \sqrt{1-e^2}$

hence, $\boxed{P^2 = \frac{4\pi^2}{G M} a^3}$ (K III)

A few more notes on angular momentum & energy:

$$E = -\frac{GMm}{2a}; \text{ depends only on } a, \text{ not on } e$$

$$L = m\sqrt{GMa(1-e^2)}; \text{ for given L, minimum E is for } e = 0$$

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}$$

$$E = E_{kin} + E_{pot} \;=\; \frac{1}{2}\langle E_{pot}\rangle = -\langle E_{kin}\rangle$$

(the Virial Theorem, we'll return to it!)

Escape velocity is when $E = 0$, which implies $v_{esc} = \sqrt{\frac{2GM}{r}} = \sqrt{2}\,v_{circ}$

Puzzle: SPACE SURVIVAL RULE #1

An astronaut is accidentally left behind the space shuttle by his careless colleagues.

Which way should he aim to catch up?

$$L = m\sqrt{GMa\left(1-e^2\right)}$$

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a} < 0$$

$$P^2 = \frac{4\pi^2}{GM}a^3$$

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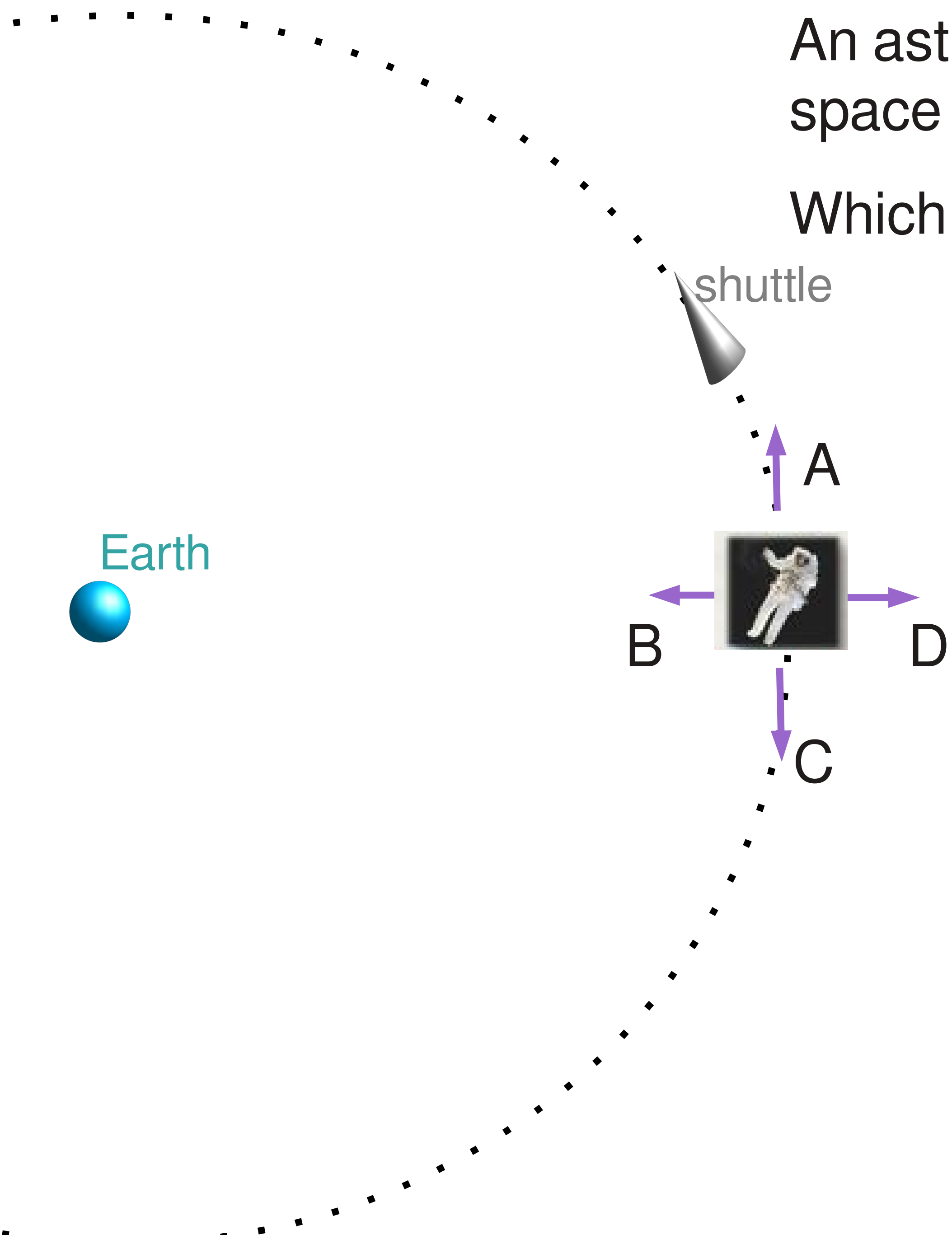

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(Tip: look up Gemini 4)