AST 1410 – Stars (2022)

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Contents

1 Syllabus

Lectures Mondays, 11-13; Wednesdays, 13-14; zoom for now

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notes pdf

Stellar astrophyiscs – the success story of 20th century astronomy – requires a synthesis of most of basic physics (thermodynamics, quantum mechanics, and nuclear physics). It underlies nearly all of astronomy, from reionisation to galaxy evolution, from interstellar matter to planets, and from supernovae and planetary nebulae to white dwarfs, neutron stars, and black holes.

In this course, we will review these successes (roughly first eight weeks) and then discuss current topics and remaining puzzles (last four weeks, detailed content depending on interest).

Underlying Physics

Master equations equilibria; timescales; mass-radius and mass-luminosity relations; Hertzbrung-Russell diagram, common threads in stellar evolution, features in stellar evolution.

Equation of state fermions and bosons; pressure and energy density; ideal gas; (completely and partially) degenerate gas; radiation pressure; Boltzman distribution; Saha equation.

Heat loss radiative diffusion; conduction; opacity sources; Schwarzschild and Ledoux criteria; mixing length theory; convective flux; stellar context for convection; semi-convection.

Energy production nuclear binding energy; Coulomb barrier; reaction channels (PP, CNO, He and beyond, D/Li burning, s-/r-/p-processes); rates and neutrinos.

Stellar Evolution Themes

Low-mass stars Hayashi track; Li burning; (former) solar neutrino problem; pressure ionization and thermal ionization; convection zone advance; rotational evolution; shell burning; core-mass radius, luminosity relations; helium core flash; thermal flashes; RGB/AGB winds; production of intermediate-mass elements; white dwarfs.

High-mass stars CNO burning and core convection; Eddington luminosity and formation/mass loss of high mass stars; nucleosynthetic yield of high mass stars; rotational evolution; feedback to the galaxy; electron-capture, core collapse and pair instability SNe; Pop III stars; neutron stars, black holes.

Binary evolution frequency of binarity; tidal synchronization and circularization; Roche lobe overflow; conservative and non-conservative mass transfer; common-envelope evolution; mergers; blue and red stragglers.

Nucleosynthesis production of the elements; explosive nucleosynthesis; r and s process; core-collapse supernovae; thermonuclear supernovae; various dredge-ups; thermal pulses.

Course texts

The main book we will use is *Stellar Structure and Evolution* (KWW; Kippenhahn, Weigert & Weiss, Springer-Verlag, 2012; the first edition, without Weiss, is OK too). Especially for those who did not take undergraduate astrophysics, I strongly recommend *An Introduction to Modern Astrophysics*, by Carroll & Ostlie (2nd edition; Cambridge University Press, 2017). This book introduces more empirical knowledge (and jargon) assumed known in KWW, and is used for the UofT undergraduate courses AST 221 and AST 320; below, I'll at times refer to relevant notes and mini problem sets from the latter.

Evaluation

- Two problem sets (30% total), due two weeks after posting. The second will use `mesa` to investigate stellar evolution.
  - Problem set 1 (pdf), due Feb 9.
  - Problem set 2 (pdf), due Mar 23.
- Presentations (30% total), one short one (10%) explaining a specific concept (8 min., plus 7 min. discussion; see list of topics) and one long one (20%) on a more advanced topic (after interest, though here are some suggestions; 20 min., plus 10 min. discussion on March 14, 21, 28).
- Final exam (40%; oral).

2 Master equations

Equilibria; timescales; mass-radius and mass-luminosity relations; Hertzbrung-Russell diagram, common threads in stellar evolution, features in stellar evolution.

Mon 10 Jan

Textbook • Read ahead: KWW 1–2.3, 3.1 [and AST320 notes 1].

Exercises
- Check wiki:HRD. Why are there limits on the left (no regular stars hotter & dimmer than the main sequence), at the top (no ultra-luminous stars), and on the right (no very cool stars)?
- What mass-radius relation would you expect for a set of stars with the same central temperature? Would more massive stars have higher or lower central density and pressure?
- AST320 mini problem sets: I (revision of what we did in class)
- Derive the Virial Theorem (assuming ideal gas), following hint below.

General knowledge questions
- Explain why we know what the Sun's central temperature roughly ought to be, and how we know what it actually is.
- Which have higher central pressure, high-mass or low-mass main-sequence stars? Roughly, what is their mass-radius relation? Derive this.
- Why is nuclear fusion stable inside a main-sequence star?

- What is a star?
  - Me: ball of gas
  - `http://en.wikipedia.org/wiki/Star`: A star is a massive, luminous sphere of plasma held together by gravity.
- Implication: P high → T high → loose heat Unless high P without high T → degeneracy (brown dwarf, degenerate helium core, white dwarf).
- Basics of evolution: contraction, heat up, fusion When does it stop?
- Virial theorem: $E_{int}=-\frac{1}{2}E_{grav}$; $E_{tot}=E_{int}+E_{grav}=\frac{1}{2}E_{grav}$
  - Roughly, one has $E_{grav} \approx GM^2/R$, $E_{int} \approx N\langle e_{int}\rangle = (M/\mu m_H)3/2\ kT$, hence $kT \approx GM\mu m_H/R$.
  - To derive it formally, multiply HE by r on both sides and integrate over sphere; use that for ideal gas $U = 3/2\ nkT = 3/2\ P$.

Wed 12 Jan

Textbook
- Covered: KWW 1–2.2, KW 5.1.1, 5.1.2 [AST 320 notes 1];
- Read independently: KWW 2.5–2.6;

- Read ahead: KW 2.3–2.4, 3.3 [AST 320 notes 2].

Exercises

- AST320 mini problem sets: VI, III

- Where do we see stars Check wiki:HRD. Why are there limits?
  - left: T_c very high $\rightarrow$ fusion (think of contraction gone too far)
  - top: L too high, matter blown away (L_{Edd})
  - right: T profile too steep $\rightarrow$ convection
- Structure equations: MC, HE, TE, EB
  - mass conservation: $\partial m/\partial r = 4\pi r^2\rho$
  - Hydrostatic equilibrium: $\partial P/\partial r = -g\rho = -Gm\rho/r^2$
  - Thermal equilibrium: $\partial T/\partial r = -(3/4ac)(\kappa\rho/T^3)(\ell/4\pi r^2)$; easiest to derive from general diffusion equation: $j = -\frac{1}{3}vl_{mfp}\nabla n$; for radiation, v=c, $l_{mfp} = 1/\sigma n = 1/\kappa\rho$, $n = aT^4$; hence, $F_{rad} = \ell/4\pi r^2 = -c/3\kappa\rho\ \partial(aT^4)/\partial r$, from which one can solve for $\partial T/\partial r$.
  - Energy balance: $\partial\ell/\partial r = 4\pi r^2\rho\epsilon$
  - To solve, needs
    * equation of state P(ρ,T,abundances)
    * opacity κ(ρ,T,abundances)
    * energy generation rate ϵ(ρ,T,abundances)
  - If T (dependence) known, can solve MC, HE separately.
  - Independent variable r or m (or P, or ...)

Mon 17 Jan

Textbook KWW 2.3, 20 [AST 320 notes 2]

Exercises

- Check you understand the basics discussed so far.
- For fun, have a look at AST320 mini problem set VII.

- Luminosity of a star
  - Simple estimates/scalings:
    * MC: $\rho \approx M/R^3$
    * HE: $P \approx GM^2/R^4$
    * TE: $L \approx acRT^4/\kappa\rho$

– Combining with ideal gas law P=($\rho/\mu m_H$)kT:
  * MC+HE: kT ≈ GMμm_H/R
  * MC+HE+TE: L ≈ acG^{4}m$_H{}^4$ μ^4M$^3/\kappa$

Note: what is radiated does **not** depend on how energy is generated; star has to provide the energy, whether by contraction or fusion.

- Homology If two stars have the same structure, i.e., m'(r')/M'=m(r)/M for all r'/R' = r/R, then:
  - MC: ρ'(r')/ρ(r) = (M'/M)(R'/R)$^{-3}$
  - HE: P'(r')/P(r) = (M'/M)^2(R'/R)$^{-4}$

  One can also derive other properties; see KWW 20.

- Real M-R and M-L relations
  - ϵ steep function of T → M/R nearly constant. Reality R ∝ M$^{0.8}$.
  - κ not constant (higher at low T, but convection moderates the effect) → L ∝ M^4
  - contribution of degeneracy → steeper at low M
  - contribution of radiation pressure → flatter at high M.

- Substantial difference: inert lump inside → shell source Or denser lump with fusion (say He core): "star inside a star"

Wed 19 Jan

Textbook KWW 2.4, 3.3, 4.4 [AST 320 notes 2], KWW 25.3 (esp. 25.3.2)

Exercises
- Calculate dynamical time for the Sun ($\langle\rho\rangle$ ≈1 g/cm^3), a neutron star (∼10^{14} g/cm^3), and the Universe as a whole (∼1 m^{-3}≈10^{-30} g/cm^3).

Exercises
- AST320 mini problem sets: II

General knowledge questions
- Describe what happens as a cloud starts to collapse and form a star. What is the difference between the collapse and contraction stages? What happens to the internal temperature in both? When does the contraction phase end, and why does the end point depend on the mass of the object?
- Please pick a topic for your short presentation!

- Stability Upon compression, $\rho \propto R^{-3}$. Thus, for an adiabatic perturbation, $P \propto \rho^\gamma \propto R^{-3\gamma}$ (with γ a suitable average over the star). To keep in HE, P should increase as R^{-4} or faster, i.e., $3\gamma>4$ or $\gamma>4/3$.
- Timescales
  - Dynamical: $\tau_{dyn} \approx 1/(G\rho)^{1/2}$ What if not in HE?
    Equation of motion: $\rho\partial v/\partial t = \rho\partial^2 r/\partial t^2 = -\nabla P + \rho\nabla\Phi = -\partial P/\partial r - \rho g$
    * Pressure drops away? $\tau_{ff} \approx (R/g)^{1/2} \approx 1/(G\rho)^{1/2}$
    * Gravity drops away? $\tau_{expl} \approx R(P/\rho)^{-1/2} \approx R/c_s \approx 1/(G\rho)^{1/2}$
  - Thermal: $t_{th} \approx E_{th}/L$ ($\approx GM^2/RL$ for whole star) If not in TE over some distance d: $F = -(vl_{mfp}/3)\nabla U \approx (vl_{mfp}/3)U/d$ (where $vl_{mfp}=c/\kappa\rho$ for diffusion by radiation).
    Hence, timescale $\tau_{adj} \approx Ud^3/Fd^2 \approx 3Ud^3/vl_{mfp}Ud \approx d^2/vl_{mfp} \approx (l_{mfp}/v)(d/l_{mfp})^2$ (random walk: t_{step} N_{steps})
    Timescale for radiative damping of pulsations? Higher order $\rightarrow$ smaller d $\rightarrow$ faster damping.
  - Nuclear: $t_{nuc} \approx E_{nuc}/L$

3 Equation of state

Fermions and bosons; pressure and energy density; ideal gas; (completely and partially) degenerate gas; radiation pressure; Boltzman distribution; Saha equation.

Mon Jan 24

Textbook KWW 19.1–19.4, 19.9 (and scan through rest except 19.11), 15 (except 15.4) [AST 320 notes 4]

Exercises
- Write your own polytrope integrator (you'll need it for the first problem set; I suggest using python; if you are clueless, have a look at my simple integrator for an isothermal atmosphere).
- Use it to calculate the radius of a star with a solar mass and with central density and pressure like the Sun, for n=1.5 and 3.
- AST320 mini problem sets: IV

Further exercises
- For classical particles, show that n(p)dp is a Maxwellian, and that one recovers the ideal gas law.

- For photons, μ=0. Show that $U_\nu d\nu$ equals the Planck functions, and that its integral equals aT^4 (note: $\int_0$^∞ dx $x^3/(\exp(x)-1)=\pi^4/15$)

General knowledge question

- What is a polytropic equation of state? Give examples of objects for which this is a very good approximation, and explain why it is.

- Polytropes: P = Kρ^γ $\equiv$ $K\rho^{1+1/n}$ For K, γ, n $\equiv$ 1/(γ-1) constant, can integrate HE+MC.

  Examples:

  - Constant density (incompressible fluid)
  - Isothermal (part)
  - Completely convective
  - Degenerate (K fixed)

  For given K, n, know ρ(r), P(r), E_g, etc.; see AST 320 notes 4, esp. Table 4.1.

- EOS: Pressure integral: P = (1/3) $\int_p n_p v_p p dp$
  - NR: $v_p p \rightarrow p^2/m = 2e_p \rightarrow P=(2/3)U \rightarrow$ Virial Theorem: E_g=-2E_i, E_{tot}=(1/2)E_g
  - ER: $v_p p \rightarrow cp = e_p \rightarrow P=(1/3)U \rightarrow$ Virial Theorem: E_g=-E_i, E_{tot}=0
  - generally, $n_p=n(e_p)g(4\pi p^2/h^3)dp$, $n(e_p)=1/[\exp((e-\mu)/kT) \pm 1]$ +1:Fermions; -1: bosons; μ: chemical potential; g: number of internal states (such as spin)
    (For a nice description of the meaning of *The elusive chemical potential* μ, see Baierlein `http://adsabs.harvard.edu/abs/2001AmJPh..69..423B`.)
  - ignore $\pm$1: classical particles $\rightarrow$ ideal gas law: $P=(\rho/\mu m_H)kT$ (μ here is mean molecular weight)
  - photons: -1, μ=0 $\rightarrow$ get BB $\rightarrow$ $P=(1/3)aT^4$
  - electrons: +1: completely degenerate $\rightarrow$ fill up to $p_F = h(3n/4\pi g)^{1/3}$
    - NRCD: $P=K_1(\rho/\mu_e m_H)^{5/3}$, $K_1=(3/4\pi g)^{2/3}(h^2/5m_e)\approx 2.34\times10^{-38}$ N m^3
    - ERCD: $P=K_2(\rho/\mu_e)^{4/3}$ $K_2=(3/4\pi g)^{1/3}(hc/4)\approx 2.45\times10^{-26}$ N m^2

- Complications: molecular/atomic/nuclear dissociation, pair formation
- Combinations
  - Simplest: whichever dominates, or at least add radiation
  - Rough estimate everywhere (Paczynski `http://adsabs.harvard.edu/abs/1983ApJ...267..315P`:
    - $\mathrm{P} = \mathrm{P_e}+\mathrm{P_i}+\mathrm{P_{rad}} = \mathrm{P_e}+(\rho/\mu_\mathrm{i}\mathrm{m_H})\mathrm{kT} + (1/3)\mathrm{aT}^4$
    - $\mathrm{P_e} = (\mathrm{P^2}_\mathrm{e,ideal}+\mathrm{P^2}_\mathrm{e,cd})^{1/2}$, $\mathrm{P_{e,ideal}}=(\rho/\mu_\mathrm{e}\mathrm{m_H})\mathrm{kT}$
    - $\mathrm{P_{e,cd}} = (\mathrm{P^{-2}}_\mathrm{e,nrcd}+\mathrm{P^{-2}}_\mathrm{e,ercd})^{-1/2}$
  - EOS from look-up table
    - for completely ionised gas: Helmholtz (Timmes+Swesty `http://adsabs.harvard.edu/abs/2000ApJS..126..501T`; includes pair formation, explanation in Timmes+Arnett `http://adsabs.harvard.edu/abs/1999ApJS..125..277T`)
    - MESA

Wed Jan 26

Textbook Discussed in class: KW 14 (part of AST 320 notes 5); not discussed but to be read: KW 4.

Exercises • AST320 mini problem sets: V

- Another way to think about ionisation, etc. (Different from how I discussed it in class, which was based on KW 14.)

  Consider a fixed volume V at a fixed temperature T (or, equivalently, constant ρ and T). In thermal equilibrium, systems go to their most probable state, i.e., one maximizes entropy, $S = k \log Z$, where Z is the partition function, a sum over all possible states i, weighted by $\exp(-E_i/kT)$. Usually, one can split contributions, e.g., for non-interacting photons, ions, and electrons, one has $Z = Z_\gamma \times Z_e \times Z_i$ (and thus $S = k \sum \log Z$).

  In the volume, for one particle at some momentum p, the number of phase space elements available is $(V/h^3) \times 4\pi p^2 dp$, with a probability $\exp(-\epsilon_p/kT)$. The total number of phase space elements is thus $\sim (V/h^3)p_{th}^3$, where p_{th} is some typical momentum associated with the temperature. Doing the integral gives the Maxwellian and $p_{th} = \sqrt{2\pi mkT}$. Maybe more insightful is follow Baierlein `http:`

//adsabs.harvard.edu/abs/2001AmJPh..69..423B and define a typical size, $\lambda_{th} \equiv h/p_{th}$, the "thermal De Broglie" wavelength. Then, the number of possible states is simply V/λ_{th}^3. For a set of N identical particles, the contribution to the partition function is thus

$$Z_N = \frac{[g(V/\lambda_{th}^3)\exp(-\epsilon/kT)]^N}{N!},$$

where g is the number of internal states, the factorial $N!$ ensures we do not overcount states where two particles are swapped, and ϵ is an energy cost beyond thermal kinetic energy there may be for having this particle.

Let's apply this to pair creation, assuming some mix of photons, ions, electrons and electron-positron pairs. Assuming a dilute plasma, their contributions to Z can be split, i.e., $Z = Z_\gamma \times Z_e \times Z_i \times Z_\pm$ (of course, the physical picture is that there is a formation rate from the interactions of two photons, balanced by an annihilation rate; for the statistics, we are only concerned about the final equilibrium). Since the electrons and positrons are independent, $Z_\pm = Z_+ \times Z_-$, with both given by the above equation with $\epsilon = m_e c^2$, but with $N_+ = N_- = N_\pm$. Hence, $Z_+ = Z_-$, and to find the number of particles, we can just find the maximum of $S_+ = k \log Z_+$, i.e.,

$$\frac{\partial S_+}{\partial N_+} = \frac{\partial k \log Z_+}{\partial N_+} = \frac{\partial}{\partial N_+} k N_+ \left[\log\left(g\frac{V}{\lambda_{th}^3}\right) - \frac{m_e c^2}{kT} - \log N_+ - 1\right] = 0,$$

where we used that for large N, $\log N! = N \log N - N$. Solving this for N_+, one finds

$$N = g\frac{V}{\lambda_{th}^3}\exp(-m_e c^2/kT).$$

Equivalently, one has $n \equiv N/V = g\exp(-m_e c^2/kT)/\lambda_{th}^3$, which has the nice implication that for classical particles, the probability for one with given internal state to exist in a given volume element λ_{th}^3 is simple $\exp(-\epsilon/kT)$. Thus, for this very small volume, the probability becomes significant for $kT \approx m_e c^2$. But when does the number of pairs become significant on larger scales? One measure to use is when $n_\pm = n_e$, i.e., when $\exp(-m_e c^2/kT) = \lambda_{th}^3 n_e/g$. For electrons ($m = m_e$), one has $\lambda_{th} = 2.4 \times 10^{-10} T_9^{-1/2}$ cm, and $n_e = \rho/\mu_e m_H = 6 \times 10^{23}(\rho_2/\mu_e)$ cm^{-3}, so it requires $T_9 \approx m_e c^2/k(11.7 + \log g T_9^{1/2}/\rho_2) \approx 0.6$, quite consistent with KW, Fig. 34.1.

One can treat ionisation similarly, writing $Z_H = Z_0 \times Z_p \times Z_e$. We need to use that $N_p = N_e = N_H - N_0$. Doing a similar derivations as above, one derives the Saha equation. Again, ionisation is well before $kT \approx \chi$. One consequence of this, is that if one, e.g., wants to know the population in excited states in hydrogen, it is easier to do this relative to the ionised state (since by the time you can excite even to the first excited state with $\epsilon_2 = \chi_H(1 - 1/4) = 10.2$ eV, hydrogen is mostly ionised). For given state s, one thus writes $n(H_0, s)/n_p = (g_s/g_p g_e n_e \lambda_{th}^3) \times \exp((\chi - \epsilon_2)/kT)$.

Finally, back to the chemical potential μ (and Baierlein `http://adsabs.harvard.edu/abs/2001AmJPh..69..423B`). In terms of above quantities, one finds $\mu = \epsilon + kT \log(g\lambda_{th}^3/n)$, but μ also enters all thermodynamic potentials (internal energy U, enthalpy H, Helmholtz free energy F, Gibbs free energy G), as an additional term $\cdots + \mu dN$, i.e., the energy required to add one particle. In particular, for constant T, V, Helmholtz is handiest: $F(T, V, N) = PV + \sum_i \mu_i N_i$ (and $dF = PdV + \sum_i \mu_i dN_i$). For pair plasma, minimizing F for $N_+ = N_-$ (holding T, V, other N constant), one requires $\mu_+ + \mu_- = 0$. With the above microscopic definition of μ, one recovers the solution. Similarly, for ionisation, $\mu_0 = \mu_p + \mu_e$. In general, for any reaction left↔right, one expects that in equilibrium, $\sum_{\rm left} \mu = \sum_{\rm right} \mu$. (In that sense, the above are missing photons – but these have $\mu_\gamma = 0$.)

All the above was for classical particles, but the same holds for non-classical ones (except of course that one cannot assume a Maxwellian once particles start to overlap, $\lambda_{th} \approx d = n^{-1/3}$). For completely degenerate neutron gas, where $\mu = \epsilon_F$, one now trivially finds that there will be a contribution of protons and electrons such that $\mu_n = \mu_p + \mu_e$. (Here, there is no μ_ν, since the neutrinos escape; for a hot proto-neutron star, where the neutrino opacity is still high, one does need to include it.) Remember, however, that above we derive a final, equilibrium state. The process to get there can be slow – not all baryons are in the form of iron yet!

4 Heat loss

Radiative diffusion; conduction; opacity sources; Schwarzschild and Ledoux criteria; mixing length theory; convective flux; stellar context for convection; semi-convection.

Mon Jan 31

Textbook Discussed in class: KWW 5.1, start of KWW 6; to be discussed as short presentations (and to be read): KWW 5.2

- Radiative flux: F_{rad}=-(1/3) ($c/\kappa\rho$) dU_{rad}/dr Like general diffusion equation: j = -(1/3) vl ∇n
- Eddington equation: dT/dr = -(3/4ac)($\kappa\rho/T^3$)(l/4πr^2)
- Rosseland mean: $1/\langle\kappa\rangle = (\pi/acT^3)\int_\nu(1/\kappa_\nu)(dB_\nu/dT)d\nu$

Wed Feb 2

Textbook KWW 6.1–6.5 (AST 320 notes 6)

Exercises
- AST320 mini problem sets: V

General knowledge question Describe the condition for a star's envelope to become convective. Why are low mass stars convective in their outer envelopes while high mass stars are convective in their inner cores?

- Criterion for convection: $-(1/\gamma)d\ln P/dr > d\ln\rho/dr$

  Schwarzschild criterion Ignoring composition gradients → $\nabla_{ad}<\nabla_{rad}$, where ∇_{ad}=(dlnT/dlnP_{ad}=1-1/γ and ∇_{rad}=(dlnT/dr)$_{rad}$/(dlnP/dr)

  Ledoux criterion With composition gradients → $\nabla_{ad}<\nabla_{rad}-f\nabla_\mu$, where ∇_μ=dlnμ/dlnP and f=($\partial\ln\rho/\partial\ln\mu$)/(-$\partial\ln\rho/\partial\ln T$); f=1 for fully-ionised ideal gas.

- Damped and driven oscillation Can be driven when the gradient is in between the Schwarzschild and Ledoux criteria; see KWW 6.2 and 6.3.

Mon Feb 7

Textbook KWW 17, 7 (AST 320 notes 5, 6)

Exercises
- Redo AST320 mini problem sets VI, and think through what changes if you assume Kramers-like opacities (see KW 17.2–3).
- AST320 mini problem sets: V

General knowledge question • Describe these important sources of stellar opacity: electron scattering, free-free, bound-free, and the H- ion.

- Opacities Discussed KWW 17, AST 320 notes 5, including why electron-scattering opacity (in area/mass) is independent of density, while most other sources scale with density.

- Convective flux Generally, one can write the flux as,

$$F_{\rm conv} = \rho \overline{v}_{\rm conv} \Delta q = \rho \overline{v}_{\rm conv} c_P \Delta T,$$

  where $\overline{v}_{\rm conv}$ is a "suitable average" of the convective velocity.

  In terms of the gradients, one finds

$$F_{\rm conv} = \rho \overline{v}_{\rm conv} c_P T \frac{\ell_{\rm mix}}{2H_P} \left(\nabla - \nabla_{\rm ad}\right),$$

  where $\ell_{\rm mix}$ is the *mixing length*, usually parametrized as a fraction of the scale height, i.e., $\ell_{\rm mix} \equiv \alpha_{\rm mix} H_P$, with $\alpha_{\rm mix}$ the *mixing length parameter*.

  The estimate of $\overline{v}_{\rm conv}$ is the tricky part. We follow the AST 320 notes and balance buoyancy ($Vg\Delta\rho = \rho V g \Delta T/T$) and friction ($-A\rho v^2$); evaluate velocity at $\ell_{\rm mix}/2$; define $V/A = \beta \ell_{\rm mix}$, where β is a shape factor; and find

$$v_{\rm conv}^2 = \frac{\beta g}{H_P} \frac{\ell_{\rm mix}^2}{2} \left(\nabla - \nabla_{\rm ad}\right).$$

  This leads to a convective flux given by

$$F_{\rm conv} = \rho c_P T \alpha_{\rm mix}^2 \sqrt{\frac{\beta g H_P}{8}} \left(\nabla - \nabla_{\rm ad}\right)^{3/2}.$$

  Fortunately, the difficulty does not matter much: in the interiors of stars, convection is so efficient that the final temperature gradient ends up being essentially the adiabatic one. This is why we can treat completely convection stars as constant-entropy polytropes. But near the atmosphere, this is no longer true.

- Scalings for conduction Yansong gave a nice qualitative introduction (see also KW 17.6). Here, a somewhat more mathematical one.

  Generally, the flux is $F = -\frac{1}{3} v l \nabla U$. It can be separated in different components. For photons, we saw $U = aT^4$, $v = c$ and $l = 1/\sigma n$

and hence one has $F = -(4ac/3)(T^3/\sigma n)\nabla T$ (where usually we write $\sigma n = \kappa\rho$, but it is easier not to do so here). Given the definition of conductivity through $F = -k\nabla T$, one infers an equivalent conductivity $k_\gamma = (4ac/3)(T^3/\sigma n)$.

For particles, $U = \frac{3}{2}nk_BT$ and thus $F = -\frac{1}{3}vln\frac{3}{2}k_B\nabla T$. Again writing $l = 1/n\sigma$, one finds $k = \frac{1}{3}\frac{2}{3}k_B(v/\sigma)$. For an ideal, completely ionised gas, $v \propto T^{1/2}$ and $\sigma \sim Z^2e^4/(kT)^2 \propto 1/T^2$. Hence, $k \propto T^{5/2}$.

For degenerate material, we should consider ions and electrons separately. The ions still have very short mean-free path, so do not contribute much. For the electrons, only a small fraction kT/E_F near the Fermi surface carries any heat, i.e., $U_e \sim n_e(kT/E_F)kT$, and thus $\nabla U \sim n_e(k_BT/E_F)\nabla T$. Furthermore, those electrons have velocity depending on density, not temperature. Their mean-free path still is $l = 1/n_i\sigma$ (n_i the ion density), but now $\sigma \sim Z^2e^4/E_F^2 \propto 1/E_F^2$, and thus $k_e \propto (v/\sigma n_i)n_e(k_BT/E_F) \propto vE_FT$. For non-relativistic electrons, $v \propto \rho^{1/3}$ and $E_F \propto \rho^{2/3}$, so $k_e \propto \rho T$. For relativistic particles, $v \to c$ and $E_F \propto \rho^{1/3}$, so $k_e \propto \rho^{1/3}T$.

Writing in terms of an equivalent opacity, $\kappa = (4ac/3)(T^3/k\rho)$, one finds for the ionised ideal gas, the opacity for electrons scales as $\kappa_e \propto T^{1/2}/\rho$, for non-relativistic degenerate electrons, $\kappa_e \propto T^2/\rho^2$, and for relativistic degenerate electrons, $\kappa_e \propto T^2/\rho^{4/3}$. Note that the photon opacity should also be affected, since photons can only interact with electrons near the Fermi surface, so $l_\gamma \sim 1/\sigma n_e(kT/E_F)$. Equivalently, one can write that the effective opacity scales as $T/E_F \propto T/\rho^{2/3}$ (non-relativistic) or $T/\rho^{1/3}$ (relativistic). At high densities, however, electron conduction will still win because of its steeper dependence on ρ.

Wed Feb 9

Textbook KWW 10, 11, and 12 for interest (AST 320 notes 9).

Exercises

- AST320 mini problem sets: XI, questions 1 and 2 (think ahead for question 3).
- Study KWW 24 (AST 320 notes 7) on the **Hayashi line**.
- For fun, you could also have a look at a paper by your instructor where the Hayashi line turned out to be important: `http://adsabs.harvard.edu/abs/2000ApJ...529..428V` (and the acknowledgement of the referee).

In class, mostly discussed how to think of AST320 mini problem set V and boundary conditions for stellar models – see KWW 11.3.

Mon Feb 14 (first half)

Textbook KWW 24 (Hayashi line), KWW 28 (pre-MS evolution)

Exercises
- Check you understand the qualitative shapes of proto-stellar tracks (KWW Fig. 28.3; AST 320 notes, Fig. 7.3).

5 Energy production

Nuclear binding energy; Coulomb barrier; reaction channels (PP, CNO, He and beyond, D/Li burning, s-/r-/p-processes); rates and neutrinos.

Mon Feb 14 (second half)

Textbook KWW 18.1

Mon Feb 28

Textbook KWW 18.2, 18.5.1 (p-p and CNO cycles). Slowness of p-p compared to Li+p and D+p due to weak reaction.

Exercises
- AST320 mini-PS XI, XII on the first stars.

- More on tunneling See `http://en.wikipedia.org/wiki/Quantum_tunneling`; `http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/barr.html`; similar to `http://en.wikipedia.org/wiki/Evanescent_field#Evanescent-wave_coupling` relevant for the two-prism example discussed in class.

  Note the link between fusion and radioactivity, which was solved by Gamov `http://adsabs.harvard.edu/abs/1928ZPhy...51..204G` (download PDF via UofT library; in German, but perhaps do-able even if you don't know it).

- EB revisted: $\partial l/\partial m = \epsilon - \epsilon_\nu + \epsilon_g$ Specify contributions from neutrino losses and contraction/expansion

Wed Mar 2

Textbook KWW 18–18.5 (including 18.3 and 18.4, not discussed in class).

Exercises
- AST320 mini-PS XII

- Temperature dependence Generally, we write the cross section $\sigma(E) = (S(E)/E)\exp(-b/\sqrt{E})$, and integrate over E to get $\langle\sigma v\rangle$, i.e.,

$$\langle\sigma v\rangle = \sqrt{\frac{8}{\pi\mu}}\left(\frac{1}{kT}\right)^{3/2}\int_E S(E)\exp(-E/kT - b/\sqrt{E})dE$$

  Normally, $S(E)$ can be taken out of the integral and one finds the Gamov peak, with height $\exp(-3E_0/kT)$, with $3E_0/kT = -19.721(\mu/m_u)^{1/3}(Z_aZ_b)^{2/3}T_7^{-1/3}$. But resonances can be important. The above holds if one's energy is in the far wing of a resonance, so that $S(E)$ indeed varies slowly. But if the resonance is inside the Gamov peak, it can dominate the energy dependence. In that case, one can consider it as a delta function, and the reaction rate will scale just with $\exp(-E_{res}/kT - b/\sqrt{E_{res}})$, i.e., the only temperature-dependent part comes from how many particles have the right energy. For this reason, the 3α reaction rate has a term with $\exp(-C/T)$ instead of $\exp(-C/T^{1/3})$.

6 Evolution of single stars

Mon Mar 7 - Main Sequence

Textbook KWW 28.1–2, 30 (29 for interest), AST 320 notes 10, 11.

Exercises
- Check that you understand the different ends of the main sequence for different masses listed in AST 320 notes 11, and how these relate to what one sees in the HRD (AST 320 notes Fig 10.2 is well worth studying in detail).

- Approach to the main sequence Generally, contract until some fusion process can provide the luminosity radiated. On the way to the main sequence, D and Li are fused, but for most stars, the first fusion stage that can hold up the contraction for a little while is the first part of the CNO cycle, where C is turned into N (see Fig. 7.3 in the AST 320 notes). Only when the C is exhausted does the star contract further until either the p-p chain or the full CNO cycle takes over.

- On the main sequence Hydrogen converted to Helium. In low-mass stars, radiative core so centre exhausts first. In more massive stars, convective core exhausts in one go, though the convection zone slowly becomes smaller during the main sequence. In detail, this depends on how convection actually happens, i.e., on overshooting and semi-convection.

  For both, the luminosity increases slightly. Qualitatively, one can understand this from the increase in mean molecular weight μ. Naively, one would expect a decrease in radius, but changes in stellar structure counteract this (i.e., the star does not change homologously). Only in the final stages does the radius descrease a little.

- End of the main sequence The core contracts and a shell around it ignites. In general, if a stable core can be formed, it will become isothermal. But there is a maximum (see KWW and AST 320 notes); beyond that the core has to contract and either ignite He fusion or become degenerate.

Wed Mar 9 - Giant Branch and Helium Flash

Textbook KWW 33.1–6, AST 320 notes 12, *low-mass giants*.

Exercises
- Check that you understand the basic differences between fusion in main-sequence and giant stars, and in degenerate cores.
- Ensure you understand why for low-mass stars, the helium flash happens at a fixed core mass and luminosity, (nearly) independent of the stellar mass.
- Study the figures with evolutionary tracks and perhaps especially Fig. 12.4 in AST 320 notes.

- Giant stars For a sufficiently dilute envelope (M small and/or R large), the properties of shell determined by the core only, as the envelope is all "far away." In particular, $\mathrm{kT}\approx\mathrm{GM_c}\mu\mathrm{m_H}/\mathrm{R_c}(\mathrm{H_P}/\mathrm{R_c})$, where the ratio of the scale height to the core radius, H_P/R_c, is constant for homologous stars.

  As a consequence, if, e.g., the core contracts, T will go up and so will the luminosity, causing the envelope to expand: *mirror principle*.

- Helium flash For low-mass stars, the degenerate helium core is at about the same temperature as the shell. Eventually, helium ignites, at a core mass of about 0.45 $M_\odot$, somewhat off centre. Since the core is degenerate, a thermonuclear runaway ensues, though it does not become dynamically unstable.

Mon Mar 14 - Intermediate Mass Giants, Blue Loops

Textbook KWW 31, AST 320 notes 12, *intermediate-mass giants*; also KWW 33.3, about the "red bump" for low-mass giants.

Exercises
- Check you understand what causes the first and second dredge up.
- To better understand the loops, read Lauterborn et al., `http://adsabs.harvard.edu/abs/1971A%26A....10...97L` (for recent discussion on blue loops, see Walmswell et al. `http://adsabs.harvard.edu/abs/2015MNRAS.447.2951W`; for more general insights, Gautchy `http://adsabs.harvard.edu/abs/2018arXiv181211864G`).
- Study the figures with evolutionary tracks and perhaps especially Fig. 31.2 in KWW.

Wed Mar 16 - Asymptotic Giant Branch: Thermal Pulses and M-L relation

Textbook 34.1-34.4 (AGB thermal pulses; core-mass lum. relation), AST 320 notes 12, *thermal pulses*

General knowledge questions
- Why is nuclear fusion stable inside a main-sequence star? Under what conditions is nuclear fusion unstable? Give examples of actual objects.
- What is Eddington's luminosity limit?

- Eddington luminosity I find it easiest to derive from force balance (which makes sense only for optically thin material above a star's photosphere):
$$F_{\rm grav} = -\frac{GM}{R^2}m = F_{\rm rad} = \frac{L}{4\pi R^2}\sigma N$$
where with $m = \rho V$ and $\sigma N = \sigma n V = \kappa\rho V$, one finds
$$L_{\rm Edd} = \frac{4\pi cGM}{\kappa}.$$

- End of the AGB Near the end of the AGB, the luminosity from the shell approaches the Eddington lumnisity relevant for electron-scattering opacity. At the cool photosphere, the opacity generally is smaller except when it gets cold enough for dust grains to form. This becomes particularly easy after C has been dredged up. (And pulsations help too.)

Mon Mar 21 - Nucleosynthesis on the AGB; symptotic Giant Branch

Textbook KWW 34.5-34.6 (nucleosynthesis; mass loss; white dwarf initial-final mass relation)

Exercises
- Read also the rest of KWW 34.

General knowledge questions
- Sketch out a Hertsprung-Russell diagram. Indicate where on the main sequence different spectral classes lie. Draw and describe the post main-sequence tracks of both low- and high-mass stars.
- The so-called r- and s- processes are mechanisms that produce elements heavier than iron. Describe these mechanisms and evidence for them from abundance patterns. Where is the r- process thought to act?

Wed Mar 23 - Overall Evolution, Supernovae

Textbook KWW 34.8, 35, 36.1, 36.3 (up to 36.3.4); AST 320 notes 13

Exercises
- Study both interior (ρ-T) and exterior (T-L) diagrammes in detail, ensuring you understand the basics. (Further nice ones in the first `MESA` paper: Paxton et al. `http://adsabs.harvard.edu/abs/2011ApJS..192....3P`.)
- AST320 mini problem sets: XIII

7 Binary evolution

Mon Mar 28 - Mass transfer: stability and effects on orbit

Most stars increase in radius as they evolve, often drastically. If in a binary, they may at some point overflow their Roche lobes, leading to mass transfer to the companion. If this is stable, mass transfer will be on the evolutionary timescale. If unstable, it can be on the dynamical or thermal timescale. Masses transfer ceases when the star stops trying to expand; in giants, this is when most of the envelope has been transferred, and the remainder becomes so tenuous that it shrinks. Thus, one generally is left with just the core of the star. This process, and variations on it, is responsible for most of the more interesting stars we observe. For a general review, see Section 3 in Van den Heuvel, `http://adsabs.harvard.edu/abs/2009ASSL..359..125V`.

- Angular momentum loss Two stars can be driven closer by angular-momentum loss. For gravitational radiation (in a circular orbit),

  $$-\frac{\dot{J}}{J} = \frac{32G^3}{5c^5}\frac{M_1M_2(M_1+M_2)}{a^4},$$

  implying a merger time of $1.05{\times}10^7\,\mathrm{yr}(M/M_\odot)^{-2/3}(\mu/M_\odot)^{-1}(P/1\,\mathrm{hr})^{8/3}$, where $\mu = M_1M_2/(M_1+M_2)$ is the reduced mass, and P the orbital period. Thus, to merge within a Hubble time requires periods less than $\sim 0.5\,\mathrm{d}$.

  For binaries with low-mass stars, angular momentum can also be lost by "magnetic braking" – a solar-like wind coupled to a magnetic field. This mechanism is usually described by semi-empirical relations, which are calibrated using the rotational evolution of single stars and using population synthesis models for binaries.

- Mass loss and tranfer Consider a star that looses or transfers mass at some rate $\dot{M}$.

  - Effect on orbit The angular momentum of an orbit is given by $J = (M_1M_2/M)\sqrt{GMa}$, and thus,

    $$\frac{\dot{J}}{J} = \frac{\dot{M}_1}{M_1} + \frac{\dot{M}_2}{M_2} - \frac{1}{2}\frac{\dot{M}}{M} + \frac{1}{2}\frac{\dot{a}}{a}$$

    With this, we can now consider several cases.

    * Conservative mass transfer Consider mass transfer from star 2 to star 1. If no mass and angular momentum is lost, then $\dot{M}_1 = -\dot{M}_2$, $\dot{M} = 0$, $\dot{J} = 0$. Thus,

      $$\frac{\dot{a}}{a} = 2\frac{M_2 - M_1}{M_1M_2}\dot{M}_2 = 2(q-1)\frac{\dot{M}_2}{M_2},$$

      where $q = M_2/M_1$ is the mass ratio between the donor (star 2) and the accretor (star 1). For donors less massive than the

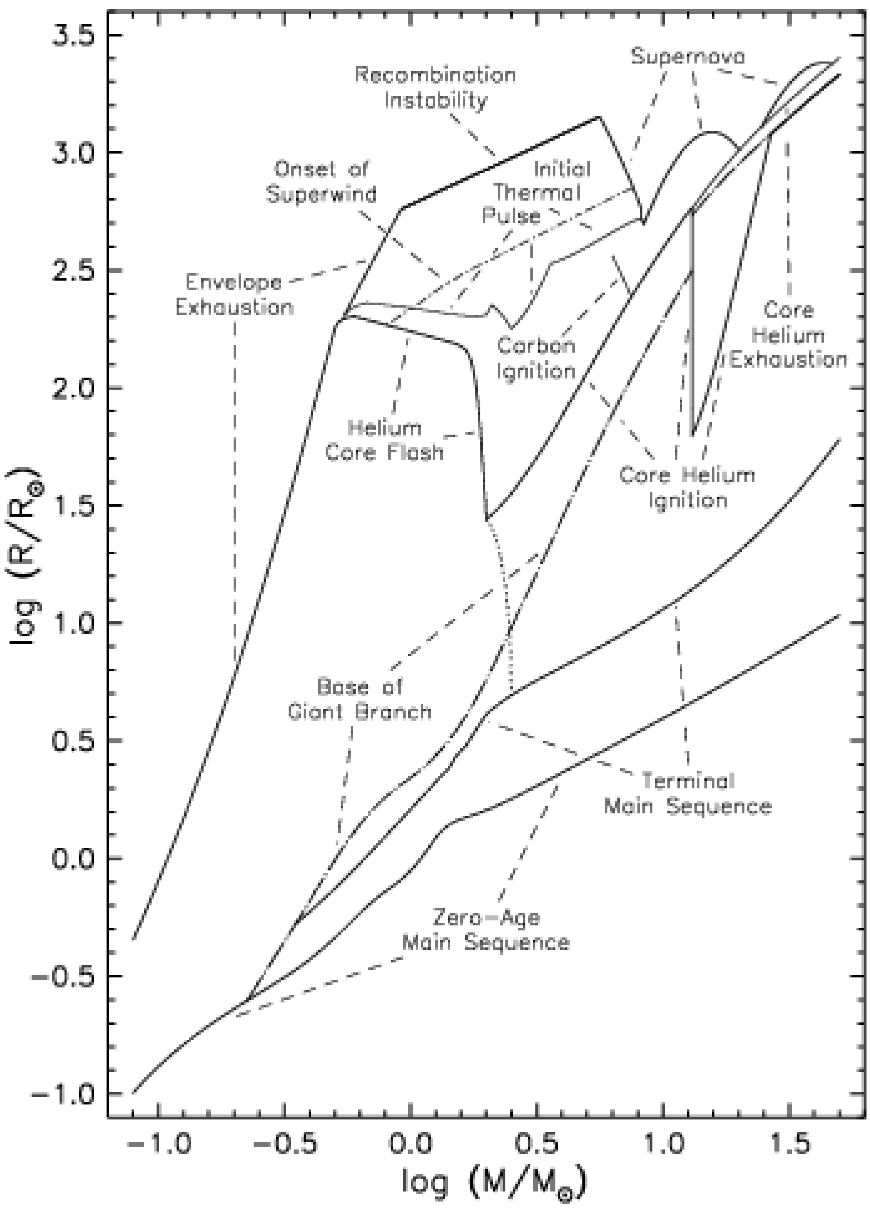

Figure 1: Radius evolution of stars of various masses. Lines indicate properties, as labelled; the one unmarked dotted line between 'helium core flash' and 'core helium ignition' marks the division between those helium cores (at lower masses) which evolve to degeneracy if stripped of their envelope, and those (at higher masses) which ignite helium non-degenerately and become helium stars. From Webbink `http://adsabs.harvard.edu/abs/2008ASSL..352..233W`, his Fig. 1.

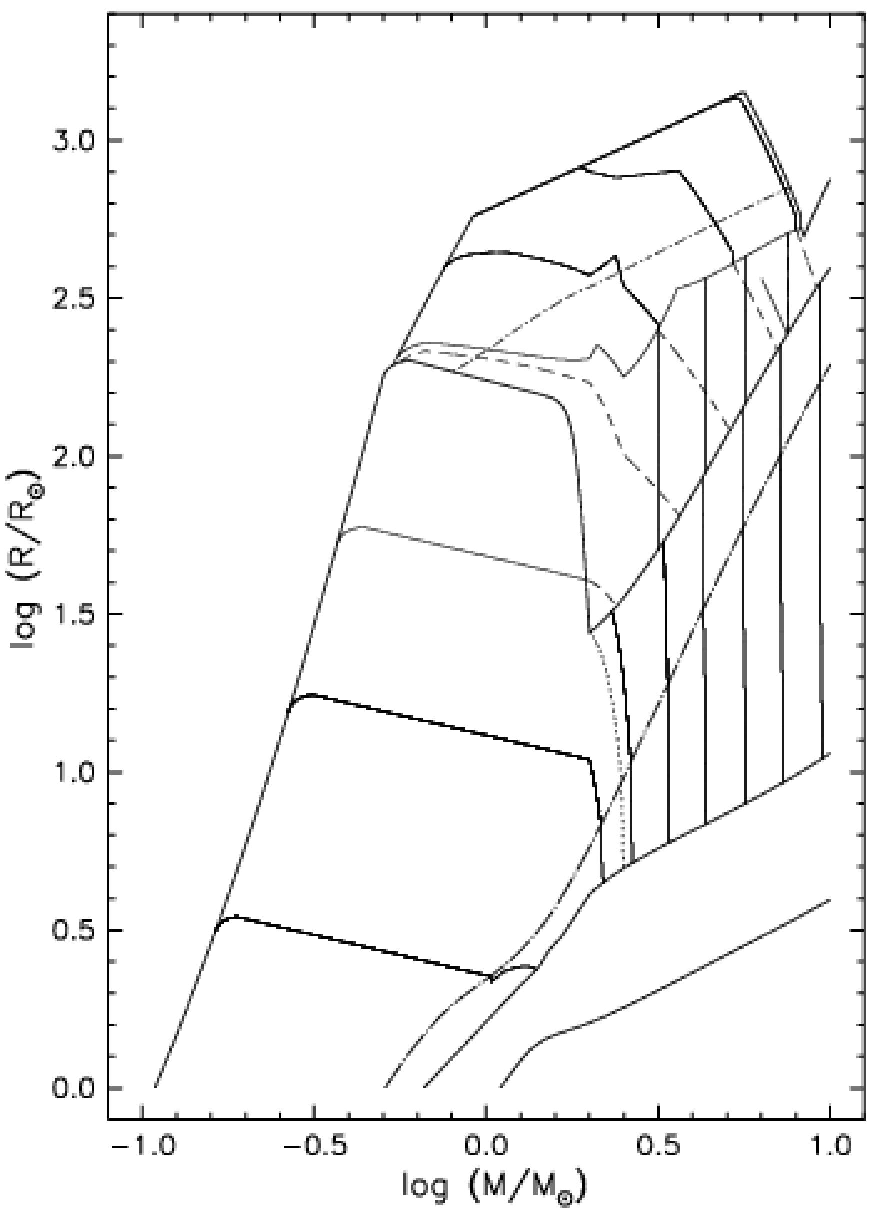

Figure 2: Core masses as a function of radius and mass. Those interior to the hydrogen-burning shell are indicated with solid lines, and dashed lines those interior to the helium-burning shell. Solid lines intersecting the base of the giant branch (dash-dotted curve) correspond to helium core masses of to 0.15, 0.25, 0.35, 0.5, 0.7, 1.0, 1.4, and 2.0 $M_\odot$; those between helium ignition and the initial thermal pulse to 0.7, 1.0, 1.4, and 2.0 $M_\odot$, and those beyond the initial thermal pulse to 0.7, 1.0, and 1.4 $M_\odot$. Dashed lines between helium ignition and initial thermal pulse correspond to carbon-oxygen core masses of 0.35, 0.5, 0.7, 1.0, and 1.4 M$_\odot$. Beyond the initial thermal pulse, helium and carbon-oxygen core masses converge, with the second dredge-up phase reducing helium core masses above about 0.8 $M_\odot$ to the carbon-oxygen core. From Webbink `http://adsabs.harvard.edu/abs/2008ASSL..352..233W`, his Fig. 2.

accretor, the orbit expands upon mass transfer (remember that $\dot{M}_2 < 0$).
Looking at the Roche lobe for a less massive donor, for which $R_L \approx 0.46a(M_2/M)^{1/3}$, one finds

$$\frac{\dot{R}_L}{R_L} = \frac{\dot{a}}{a} + \frac{1}{3}\frac{\dot{M}_2}{M_2} = 2\left(q - \frac{5}{6}\right)\frac{\dot{M}_2}{M_2},$$

showing that the Roche lobe, as expected, grows a little slower than the orbital separation. (An analysis valid for all q would use the approximation of Eggleton `http://adsabs.harvard.edu/abs/1983ApJ...268..368E`, $R_L/a \simeq 0.46q^{2/3}/[0.6q^{2/3} + \ln(1+q^{1/3})]$.)

* Spherically symmetric wind $\dot{M}_2 = \dot{M}$, $\dot{M}_1 = 0$, $\dot{J} = (\dot{M}_2/M_2)(M_1/M)J$. Hence,

$$\frac{\dot{a}}{a} = 2\left(\frac{M_1\dot{M}}{M_2M} - \frac{\dot{M}}{M_2} + \frac{\dot{M}}{2M}\right) = -\frac{\dot{M}}{M}.$$

Thus, for mass loss ($\dot{M} < 0$), the orbit expands.

* Spherically re-emitted wind $\dot{M}_2 = \dot{M}$, $\dot{M}_1 = 0$, $\dot{J} = (\dot{M}_2/M_1)(M_2/M)J$ (idea is that accretor cannot handle mass transferred to it and re-emits it as a wind). Hence,

$$\frac{\dot{a}}{a} = 2\left(\frac{M_2\dot{M}}{M_1M} - \frac{\dot{M}}{M_2} + \frac{\dot{M}}{2M}\right) = \frac{2q^2 - 2 - q}{1+q}\frac{\dot{M}}{M}.$$

Hence, orbit expands for $q < (1+\sqrt{17})/4 = 1.28$ (with again a somewhat lower value for increasing Roche-lobe radius), i.e., it is less quickly unstable than for conservative mass transfer. For a more detailed analysis, see Soberman et al., `http://adsabs.harvard.edu/abs/1997A&A...327..620S`

– Effect on stellar radius If the mass is lost from the outside of a star, the star becomes initially smaller, but on a hydrodynamic timescale it will partially re-expand in responds to the decreased pressure. Which effect dominates depends on the internal structure of the star. Generally, for thermal envelopes, the stars shrinks inside its Roche lobe, re-expanding only on the thermal timescale, typically to nearly its original size (especially for giants). However, a complication for thermal-timescale mass transfer is that, if the secondary is substantially less massive, it cannot accrete sufficiently fast and will bloat itself. For massive stars, this leads to contact, and almost certainly further mass loss and/or a merger. If this can be avoided, then eventually the two stars have equal mass, after which further mass transfer leads to expansion of the orbit, and eventually the donor will regain thermal equilibrium. After that, any further mass transfer is on its evolutionary timescale.

Completely convective stars, or stars with deep convective layers, however, increase in size upon mass loss. For completely convective stars, which are described well by polytropes with $P = K\rho^\gamma$ with $\gamma = \frac{5}{3}$ (and thus $n = 1.5$), this follows immediately from the mass radius relation: $R \propto M^{-1/3}$ (true for constant K, i.e., for constant entropy or completely degenerate, non-relativistic gas). Comparing this to the change in Roche lobe for conservative mass transfer, one sees that stability requires that

$$2\left(q - \frac{5}{6}\right) < -\frac{1}{3} \Leftrightarrow q < \frac{2}{3} \qquad \text{for} \quad n = 1.5.$$

Wed Mar 30 - Common envelope evolution

When dynamically unstable mass transfer starts, the stars enter a common envelope. This will lead to a merger unless one envelope is relatively loosely bound, e.g., if the donor is a red giant. The process is still very uncertain, and usually an energy criterion is used to decide whether or not a complete merger occurs. We write the initial orbital energy as $E_{\rm orb,i} = GM_1M_2/2a_{\rm i}$, the final one as $E_{\rm orb,f} = GM_{\rm 1,c}M_2/2a_{\rm f}$, and the envelope binding energy as $E_{\rm e} = GM_1M_{\rm 1,e}/\lambda R_1$. Taking $M_{\rm 1,e} = M_1 - M_{\rm 1,c}$, a roche-lobe filling star ($R_1 = R_{\rm L}$), and assuming an efficiency $\alpha_{\rm CE} = E_{\rm e}/(E_{\rm orb,f} - E_{\rm orb,i})$, one finds a total shrinkage of the orbit,

$$\frac{a_{\rm f}}{a_{\rm i}} = \frac{M_{\rm 1,c}}{M_1}\left[1 + \frac{2}{\alpha_{\rm CE}\lambda}\frac{a_{\rm i}}{R_{\rm L}}\frac{M_1 - M_{\rm 1,c}}{M_2}\right]^{-1}$$

This shrinkage is usually very large.

It has been tried to calibrate this using systems in which the donor was a red giant and hence its leftover a helium white dwarf. In this case, we know from the relation between core-mass and radius what the initial separation was, so we can try to calibrate the efficiency. Tracing back the evolution

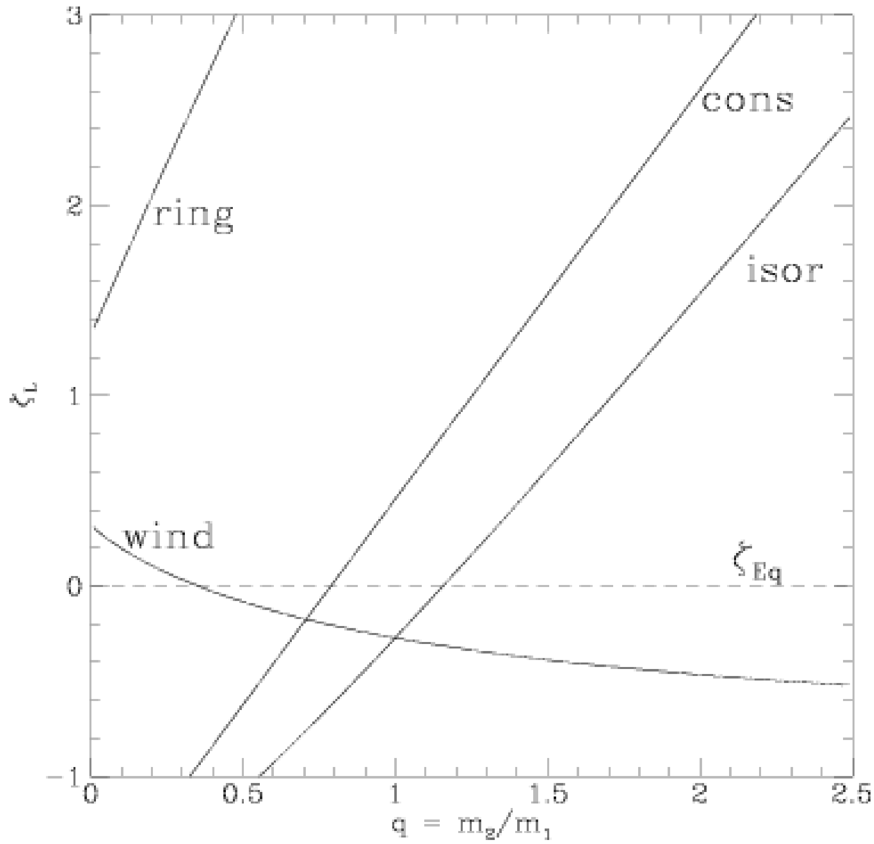

Figure 3: $\zeta_{\rm L} \equiv \partial \ln R_{\rm L}/\partial \ln M_2$ as a function of mass ratio, with all mass transfer through a single channel: conservative (cons); isotropic wind from donor star (wind); isotropic re-emission of matter, from vicinity of 'accreting' star (iso-r). (Also shown is a ring formation, indicative of mass loss from an outer Lagrange point). From Soberman et al., `http://adsabs.harvard.edu/abs/1997A&A...327..620S`, their Fig. 4.

of double helium white dwarfs, however, Nelemans et al. http://adsabs.harvard.edu/abs/2000A%26A...360.1011N found that it cannot hold for the first mass-transfer phase. They proposed an alternative description based on angular momentum loss, but this was criticised strongly (e.g., Webbink, http://adsabs.harvard.edu/abs/2008ASSL..352..233W, which also is a great review of common-envelope evolution). Still, the conclusion stands that for not too extreme mass ratios, mass transfer apparently is stabilised somehow.