Mini-Problem Set I: Basics of a star's life

due 10 Jan 2014

Consider a star with a mass like that of the Sun, but starting with a larger radius and a temperature too cool for nuclear fusion.

1. What happens to the total, potential, and kinetic energy of the star as it radiates into space?
2. As a result, how do you expect the radius and internal temperature to change?
3. When will the star stabilise?

Mini-Problem Set II: Halting the collapse of a cloud *due 13 Jan 2014*

Consider a cloud that has just started to collapse.

1. Use scaling arguments to show that for an isothermal gas, once collapse starts the gravitational term in the equation of hydrostatic equilibrium increases more rapidly than the pressure gradient, so that the free-fall approximation is reasonable.
2. What happens once the gas changes from isothermal to adiabatic? Does one indeed expect the collapse to halt eventually for this case?
3. Suppose the cloud originally was permeated by a magnetic field, with magnetic pressure $\frac{1}{8\pi}B^2$ below the gas pressure? Could the increase in magnetic field strength halt the collapse? *Hint: assume conservation of magnetic flux, BR^2.*
4. Suppose the cloud originally were rotating slowly (with a velocity below the Keplerian one). Could the spin-up during contraction halt the collapse? *Hint: assume conservation of angular momentum.*

Mini-Problem Set III: Equation of state in the cores of stars *due 17 Jan 2014*

What processes dominate the pressure in cores of stars? We use that due to the high temperature sensitivity of nuclear fusion, stars on the main sequence all have rather similar central temperatures.

1. Show that given a (roughly) similar temperature on the main sequence, radius is (roughly) proportional to mass. Given this, do central pressure and density increase or decrease with mass? How do they scale?
2. Look up the temperature, density, and pressure in the core of the Sun in CO. For the Sun, does radiation pressure make an important contribution to the total pressure? Given the scalings you derived above, for what mass does radiation pressure become important?
3. Another source of pressure, which we will discuss in class, is degeneracy pressure. To estimate whether it is important in the Sun, use Heisenberg's uncertainty relation to estimate what momentum electrons would have just due to the fact that they are close together (and hence we 'know' their location to some precision; see Co, Eq. 5.19; also, end of p. 567). To what kinetic energy does this momentum correspond? How does this compare with the thermal energy, kT? Given the scalings you derived above, for what mass does degeneracy become important?

For presentation (and further thought): What would change for stars made of pure helium instead of solar composition? *Note: helium fusion requires* $T \simeq 10^8$ *K.*

Mini-Problem Set IV: Basics of degenerate objects *due 20 Jan 2014*

1. Write down the general scaling relations for central pressure and density as a function of mass and radius. For all stars having the same polytropic equation of state, the constant terms will be the same. Use this to derive how radius scales with mass for stars composed of NRCD gas.
2. And how does radius scale with mass for ERCD gas? What does your result mean?
3. Describe briefly in physical terms (e.g., particle momenta and velocities) what happens if a degenerate object (such as a white dwarf) becomes more and more massive (e.g., by accretion of matter).
4. Before an object becomes extremely relativistic, nuclear fusion may well start. To show what would happen, draw an EOS diagramme such as Fig. 3.2 (but going to higher temperature) and sketch how central density and temperature would evolve. Assume an initial density of $10^6\,\mathrm{g\,cm^{-3}}$ and initial temperature of $10^6\,\mathrm{K}$.

For presentation (and further thought): What would be required for a white dwarf to explode (e.g., as a type Ia supernova)? Approaching very high density, pressure support would be lost because of "inverse beta decay" (see Fig. 4.2 and its caption). Why is this?

Mini-Problem Set V: Completely convective stars *due 27 Jan 2014*

Stars with masses $\lesssim 0.3\,M_\odot$ are completely convective, and a polytropic model should work fairly well. We will check this using the models shown in Fig. 3.3. Note: since we determine how well it works, keep a few significant digits in the calculations and read off the figure carefully (especially for density)!

1. Use the appropriate polytropic model to calculate central density and pressure for a completely convective $0.30\,M_\odot$ star with a $0.33\,R_\odot$ radius. Infer the central temperature, assuming an ideal gas with the composition indicated in Fig. 3.3. Repeat for a $0.10\,M_\odot$ star with $0.13\,R_\odot$ radius.
2. Compare your results with the central density and temperature shown in Fig. 3.3 for the two masses. You will find not all results are consistent. What assumption might be wrong? *(For the consistency check, assume a 10% uncertainty in radius – Pols et al. do not give radii and various theoretical models differ by this amount.)*
3. What is the expected slope in the temperature, density diagramme for $n = 1.5$? Make a copy of Fig. 3.3, and sketch a line with that slope starting at the centre for the $0.3\,M_\odot$ star. Does it match the straight part of the model curve?
4. For $1\,M_\odot$, the central density and temperature should not match those expected for a completely convective star as closely. Is this true? (Assume $R = 1\,R_\odot$.) What polytropic index n would give the best match? *(For the astute: you may notice that the central density in Fig. 3.3 is much lower than for the current Sun – this is because it is for a zero-age main-sequence star.)*
5. For the best estimate of n you found, sketch a line with the corresponding slope in the copy you made of Fig. 3.3, starting at the centre for the $1\,M_\odot$ star. Does it roughly match the model? Below $\log T \simeq 6.5$, the model track becomes much steeper, and runs parallel to the track for the $0.3\,M_\odot$ star. Which part of the Sun do you think this is?

For presentation (and further thought): Why do the low-mass curves have wiggles coincident with ionisation and dissociation zones? What polytropic index would be consistent with the slope seen for the 10 and $100\,M_\odot$ models? Compare with what is expected for the "Eddington Standard Model" (CO, p.339ff).

Mini-Problem Set VI: The luminosity of a star *due 31 Jan 2014*

Consider a star in hydrostatic equilibrium in which energy transport is by radiation.

1. Use the equations of radiative energy transport and hydrostatic equilibrium to derive two scaling relations for $T_{\rm c}$ in terms of other stellar properties (here, assume the gas is described by the ideal gas law, and that the opacity is independent of density and temperature). Combine the two to derive how L scales with M and R.
2. Why is your result independent of the source of energy?

For presentation (and further thought): For more massive stars, radiation pressure becomes important. How does this affect the scaling? And what is the effect of composition? Would helium stars be brighter or fainter than ones composed of mostly hydrogen? Roughly by how much?

Mini-Problem Set VII: Fully ionised atmospheres

due 3 Feb 2014

Consider hydrostatic equilibrium in the atmosphere of a hot star.

1. Assume a B0 star. Look up its properties in Carroll & Ostlie, and calculate the gravity g and escape velocity $v_{\rm esc}$, as well as the typical thermal velocities $v_{\rm p}$ and $v_{\rm e}$ for protons and electrons. Considering only gravity, are the protons bound? What about the electrons?
2. Generally, the difference in mass between electrons and protons implies the existence of an electric field in the atmosphere, which exerts a force eZE. What electric field strength eE is needed for equilibrium? Express forces using units of $m_{\rm p}g$, and assume an atmosphere in hydrostatic equilibrium composed of pure, fully ionised hydrogen, ignore the electron mass, and use that the force on each particle due to the pressure gradient is equal to $\frac{1}{n}\frac{{\rm d}P}{{\rm d}r}$ (where n is the particle number density).
3. Now suppose that a single helium nucleus would be placed in this atmosphere. What net force would it feel?

For presentation (and further thought): Do you think the above could affect how we see different astronomical objects? Which ones most? Can you estimate the speed with which the He would sink?

Mini-Problem Set VIII: Dating stars using Lithium: contraction *due 7 Feb 2014*

As pre-main-sequence stars contract, at some point their internal temperature becomes sufficiently hot that Lithium is destroyed. This will occur earlier for more massive stars, and hence one can estimate an age for a group of stars by finding the highest-mass star that still has Lithium. In this problem set, we will determine how stars contract and in the next two we will use this to derive an analytical relation between this critical mass and age, following Bildsten et al. (1997, ApJ 482, 442 [you do not need the reference, but feel free to look it up if you want to]). We will make two assumptions: (i) that the stars are completely convective and can be described by the appropriate, $n = 1.5$ polytrope; and (ii) that the surface temperature remains constant during contraction.

1. For a $n = 1.5$ polytope, what is the total (potential plus kinetic) energy of the star? Given an effective temperature $T_{\rm eff}$ and radius R, you know the luminosity; assuming this is powered by contraction, what is the contraction rate $\mathrm{d}R/\mathrm{d}t$? (Write in terms of M, R, and $T_{\rm eff}$, keeping all constants.)
2. Solve the differential equation, using that $R \to \infty$ as $t \to 0$ for the integration constant. Inserting physical constants, and scaling to useful numbers, you should find $R = 0.85\,R_\odot\,(M/0.1\,M_\odot)^{2/3}(T_{\rm eff}/3000\,\mathrm{K})^{-4/3}(t/1\,\mathrm{Myr})^{-1/3}$.
3. Again use the fact that the star can be described by a $n = 1.5$ polytrope, as well as the ideal gas law, to find the central temperature in terms of M and R. Insert the relation for the radius above, to show that $T_{\rm c} = 0.88 \times 10^6\,\mathrm{K}\,(M/0.1\,M_\odot)^{1/3}(\mu/0.6)(T_{\rm eff}/3000\,\mathrm{K})^{4/3}(t/1\,\mathrm{Myr})^{1/3}$.
4. We will turn to Lithium burning in the next mini problem set. Here, end with calculating the time it takes for a $0.1\,M_\odot$ star to reach the main sequence ($T_{\rm c} = 4.3 \times 10^6\,\mathrm{K}$; Fig. 3.3). How does this compare with the lifetime of massive stars?

For presentation (and further thought): How would a radiative stars, well-described by the Eddington model (mini-PS VI) contract? In Fig. 7.3, you see that more massive stars initially contract at roughly constant temperature and decreasing luminosity, but later show increasing temperature and (slightly) increasing luminosity. Why is this?

Mini-Problem Set IX: Dating stars using Lithium: Scalings *due 10 Feb 2014*

We continue with the previous problem set, now estimating the rate at which Lithium depletes, and how this rate scales with mass and time. In the next problem set, we will calculate actual numbers. Lithium is destroyed by fusion with a proton: ${}^{7}\mathrm{Li} + \mathrm{p} \rightarrow 2\,{}^{4}\mathrm{He}$. For our scaling estimate, we will use that the velocity-averaged interaction cross-section $\langle\sigma v\rangle$ is an extremely steep function of temperature, and we approximate it with a power law of the form $\langle\sigma v\rangle_{\mathrm{r}}(T/T_{\mathrm{r}})^{\nu}$ around a reference temperature T_{r}.

1. The total number of Lithium atoms will decline as $\mathrm{d}N_{\mathrm{Li}}/\mathrm{d}t = -\int_0^R 4\pi r^2 n_{\mathrm{Li}} n_{\mathrm{H}} \langle\sigma v\rangle \mathrm{d}r$. Show that using the power-law approximation for $\langle\sigma v\rangle$, expressing variables in terms of T, ρ, X and X_{Li}, and transforming r, ρ, and T to the polytropic variables ξ and θ (assuming ideal gas), this reduces to
$$\frac{1}{X_{\mathrm{Li}}}\frac{\mathrm{d}X_{\mathrm{Li}}}{\mathrm{d}t} = -\frac{3(\rho_{\mathrm{c}}/\overline{\rho})^2}{\xi_1^3}\frac{X\overline{\rho}}{m_{\mathrm{H}}}\langle\sigma v\rangle_{\mathrm{r}}\left(\frac{T_{\mathrm{c}}}{T_{\mathrm{r}}}\right)^{\nu}\int_0^{\xi_1}\xi^2\theta^{2n+\nu}\mathrm{d}\xi. \tag{IX.1}$$
2. Use the results from the previous problem set for R and T_{c} to write how $\mathrm{d}\ln X_{\mathrm{Li}}/\mathrm{d}t$ scales with M and t.
3. Consider two stars that differ in mass by a factor two, how much sooner would the more massive star start depleting Lithium for the case that $\nu \rightarrow \infty$? And what would it be for $\nu \rightarrow 0$? Discuss your result in physical terms (in a sentence or two).
4. Suppose Lithium was depleted by a factor 2 at some given age. How much longer would it take for it to be depleted by a factor 20? Again consider both the case of $\nu \rightarrow \infty$ and $\nu \rightarrow 0$ and briefly discuss your results in physical terms.

For presentation (and further thought): Slightly jumping ahead: stars with masses below $0.06\,M_{\odot}$ do not destroy Lithium, while stars with masses above $\sim 0.5\,M_{\odot}$ destroy Lithium only on the main-sequence. Why would this be?

Mini-Problem Set X: Dating stars using Lithium: Burning *due 14 Feb 2014*

We use the results from the previous two problem sets to calculate actual times at which Lithium is depleted. We use numbers from, and compare with results of, Bildsten et al. (1997, ApJ 482, 442 [as before, you do not need the reference, but feel free to look it up if you want to]).

1. We start with getting numbers for the power-law approximation to the reaction rate. Bildsten et al. give that $N_{\rm A}\langle\sigma v\rangle = S_0 f_{\rm scr} T_6^{-2/3}\exp(-aT_6^{-1/3})$, where $N_{\rm A}\equiv 1/m_{\rm H}$, $T_6\equiv T/10^6\,{\rm K}$ and $S_0 = 7.2\times10^7\,{\rm m^3\,s^{-1}\,kg^{-1}}$, $f_{\rm scr}\simeq 1$, and $a = 84.72$ (we will discuss the meaning of these numbers – and how to calculate a – when we discuss nuclear fusion). For a reference temperature $T_{\rm r} = 3\times10^6\,{\rm K}$, show that, rounded to the nearest integer value, $\nu\equiv {\rm d}\ln\langle\sigma v\rangle/{\rm d}\ln T = 19$. What is $N_{\rm A}\langle\sigma v\rangle_{\rm r}$? Check the power-law approximation at 2.5 and $3.5\times10^6\,{\rm K}$.
2. For $\nu = 19$ and $n = 1.5$, the integral over ξ in the previous problem set equals approximately 0.05. Use this, $X = 0.7$, the numbers from item 1 and the results from mini problem set VIII for R and $T_{\rm c}$ to write ${\rm d}\ln X_{\rm Li}/{\rm d}t$ in terms of M, $T_{\rm eff}$, μ, and t (scaling to $0.1\,M_\odot$, 3000 K, 0.6, and 1 Myr, respectively). Integrate this equation, and determine the time required to reduce $X_{\rm Li}$ by a factor 2. Compare both your numerical value and your exponents on M, $T_{\rm eff}$, and μ with the numerical result of Bildsten et al., $t_{\rm depl} = 50.7\,{\rm Myr}(M/0.1\,M_\odot)^{-0.663}(T_{\rm eff}/3000\,{\rm K})^{-3.50}(\mu/0.6)^{-2.09}$. How much longer is the time it takes for Lithium to be reduced by a factor 20 (i.e., an additional factor 10)?
3. Jeffries and Oliviera (2005, MNRAS 358, 13) find that in NGC 2457, stars with mass less than $0.17\,M_\odot$ still have Lithium, and they infer an age of $\sim$34 Myr. Does this match what you would infer? (*Note: if you were unsure of your own time scale, use that of Bildsten et al.*).

For presentation (and further thought): What fraction of the total luminosity will be produced by lithium burning? If you take the proton-proton cycle instead of lithium burning, what luminosity do you get for a $0.3\,M_\odot$ zero-age main-sequence star? (Which is completely convective – see mini-PS V; see there and Fig. 3.3 for basic properties.)

Mini-Problem Set XI: The First Stars *due 7 Mar 2014*

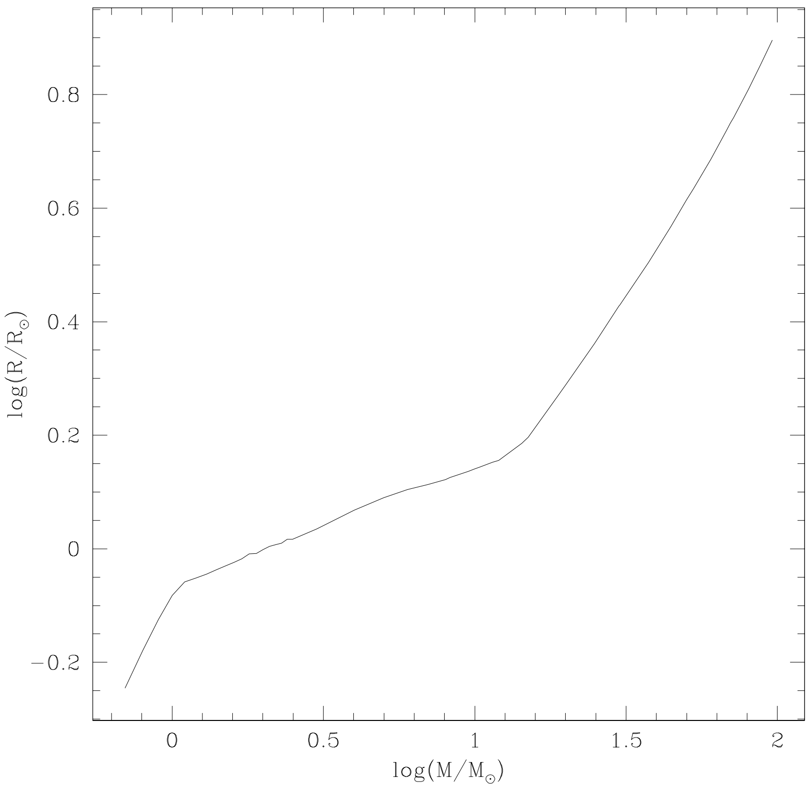
We consider the differences between the first stars ($X = 0.77$, $Y = 0.23$, $Z = 0$), and current ones (solar abundances: $X = 0.708$, $Y = 0.273$, and $Z = 0.019$).

1. Show that the equation of state is not influenced much by the differences in abundances. Consider separately ideal gas, degenerate matter, and radiation-dominated matter.
2. Discuss the differences in opacity between the two cases, considering separately electron scattering, free-free and bound-free opacities. *(NOTE: the CO expression for free-free Kramers opacity is wrong; use that on the handout.)* For main-sequence stars, will the differences affect high-mass or low-mass stars most? In what way? (i.e., how does the difference between zero and solar metallicity affect stellar properties such as luminosity, radius, convection zones? For luminosity, it may help to look back to problem set VI and forward to §10 of the notes.)
3. Which fusion processes are available in solar-abundance and zero-metallicity stars? Again discuss whether the differences affect high-mass or low-mass stars most, and in what way, therefore, zero-metallicity stars will be different from solar-metallicity ones.

Mini-Problem Set XII: Properties of the First Stars *due 10 Mar 2014*

Having considered the differences between the first stars and current ones, we continue by seeing if we can understand the mass-radius relation shown in the Figure, which was found from detailed models.

1. Assume a general mass-luminosity law $L \propto M^3$ and assume that fusion occurs in a fixed mass fraction of the star. Derive a scaling relation for L from the energy balance equation (with the power-law approximation for the energy generation rate, $\epsilon \propto \rho T^\nu$) and use it to show that $R \propto M^{(\nu-1)/(\nu+3)}$. Insert values for ν appropriate for the p-p and CNO cycles. Compare this to the parts between 1 and $13\,M_\odot$ and above $13\,M_\odot$ in the Figure. Which result is expected and which is surprising?
2. Use your result to determine the dependence of T_c on mass. Again insert values of ν, and scale to solar units by using an educated (from the previous assignment) guess for T_c for a $1\,M_\odot$ zero-metallicity star.
3. Why is there a bend in the mass-radius relation at $M \simeq 13\,M_\odot$? (Address not only why there is a bend, but also show that it makes sense it happens at approximately this mass.)



Mass-radius relation for zero-metallicity stars. Data from Marigo et al., 2001, A&A 371, 152.

For presentation (and further thought): What could cause the downturn in the mass-radius relation at masses less than $\sim 1\,M_\odot$? Considering the way stars are formed, would you expect the first stars to be typically more or less massive than current ones?

Mini-Problem Set XIII: Neutrinos from Supernovae *due 14 Mar 2014*

After the collapse of a stellar core, a proto-neutron star is formed, which is in hydrostatic equilibrium. We will assume it has mass $1.4\,M_\odot$ and radius 20 km, and use that models give a temperature between $kT = 20$ and 50 MeV.

1. Show that the temperature is (roughly) consistent with the Virial Theorem.
2. Neutrinos with a thermal distribution have average energy $\overline{E}_\nu = 3kT$, and the interaction cross-section with nucleons is roughly $\sigma(E_\nu) \sim 4 \times 10^{-46}(E_\nu/10\text{ MeV})^2\text{ m}^2$. Calculate the mean-free path inside the proto-neutron star (assuming constant density, and $kT = 35$ MeV), and estimate the time it requires neutrinos to diffuse out.
3. Use the thermal content and the timescale from above to calculate the neutrino luminosity. From the luminosity and the radius, calculate an "effective temperature" (here we'll ignore differences between photons and neutrinos). Use the temperature to estimate the average energy of emitted neutrinos and show that the neutrino emission rate is $\sim 1.3 \times 10^{57}\,\text{s}^{-1}$.
4. We now apply our results to SN 1987A, which is the most recent nearby supernova explosion: it took place in the Large Magellanic Cloud, at 50 kpc. Given the neutrino emission rate derived above, what is the neutrino flux on Earth? Use an assumed duration of 10 seconds (which takes into account the shrinking phase) to calculate the expected number detected in Kamiokande (3000 metric tonnes of ultra-pure water, detects interactions with electrons, cross section per electron $\sim 10^{-47}\,\text{m}^2$ for $E_\nu = 10$ MeV). Does it make sense that eleven events were actually detected?

For presentation (and further thought): Neutrinos have non-zero mass. Given that, would you expect neutrinos with higher energy to arrive earlier or later? Given that all events in Kamiokande arrived within 12 seconds and had energies between 6 and 20 MeV, what is the upper limit to the neutrino mass? *Hint: use that for ultra-relativistic particles ($\gamma \gg 1$), one has $c/v \simeq 1+1/2\gamma^2$ (you can derive this from CO, Eq. 4.46).*
Last year there was a claim that neutrinos travelled faster than light. Given that the SN 1987A neutrinos were detected only a few hours before the supernova was discovered in visible light, what limit can one set on their speed?