Matching Radiation-Dominated and Matter-Dominated Einstein-de Sitter Universes and an Application for Primordial Black Holes in Evolving Cosmological Backgrounds

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Abstract By matching across a surface of constant time, it is demonstrated that the spacetime for a radiation-dominated Einstein-de Sitter universe can be directly matched to the spacetime for a matter-dominated Einstein-de Sitter universe. Thus, this can serve as a model of a universe filled with radiation that suddenly is converted to matter and antimatter, or a universe filled with matter and antimatter that suddenly annihilates to leave radiation. This matching is shown to hold for asymptotically Einstein-de Sitter cosmological black hole spacetimes, yielding simplistic models of primordial black holes that evolve between being in radiation-dominated universes and matter-dominated universes.

Keywords Einstein-de Sitter universes, Spacetime matchings, Cosmological black holes, Primordial black holes

1 Introduction

The existence of the Cosmic Microwave Background (CMB) suggests the Universe was governed by radiation for most of the first ∼100,000 years. After that the radiation became redshifted relative to the matter enough that the energy of the matter began to dominate the Universe's evolution. Also, it is theorized that in the first instant of the Universe there were times when the annihilation of matter and antimatter generated photons, and when photons underwent pair production, so energy changed forms between radiation and matter as well. Thus, a realistic cosmological model should contain both radiation and matter and allow for the energy contained in each form to be able to vary in time.

Previously, models that continuously vary between radiation domination and matter domination have been studied. These models have been obtained using either of two approaches. Equations of state can be specified separately for the radiation and the matter, and an ordinary differential equation can be solved to obtain how the scale factor for the universe evolves with time. Alternatively, a scale factor can be specified that changes its evolution with time, and the energy-momentum tensor can be calculated to see what sort of variation between radiation and matter the scale factor corresponds to physically. As an example of the first approach, Jacobs [1] obtained a simple model of a universe that smoothly evolves from radiation domination to matter domination by taking the energy density of the radiation to evolve with the scale factor R as R^{-4} while the energy density of the matter evolves as R^{-3}. This realistically models the way the Universe would have evolved from radiation domination to matter domination, although it also never allows for any interchange of energy between radiation and matter whatsoever. As an example of the second approach, Coley [2] suggested a scale factor that evolves with time as

$$R(t) = t^{1/2}(1 + ht^{1/(6b)})^b \tag{1}$$

so that at small times it evolves like $t^{1/2}$, as a radiation-dominated Einstein-de Sitter universe would, and at later times it evolves like $h^b t^{2/3}$, as a matter-dominated Einstein-de Sitter universe would. Using this scale factor in the Robertson-Walker metric and interpreting the energy-momentum tensor, then depending on the value of b, Coley found that the rate of energy transfer between radiation and matter varies and can change sign, so different models of energy transfer between radiation and matter can be obtained by adjusting b.

It is unknown whether it is possible to match regions of spacetime that suddenly change between radiation domination and matter domination, instantly changing from evolving as $t^{1/2}$ to $t^{2/3}$. Previously no one appears to have attempted to do this. Presumably this is because homogeneous cosmologies are usually studied for which it is easily possible to use two-fluid models, and there has been more interest in modelling the gradual change from radiation domination to matter domination (as the CMB suggests occurred in the real Universe). It is possible to directly match spacetimes using the Darmois [3] conditions. These conditions require that the first fundamental form

$$\Omega_{\alpha\beta} = g_{ab}\frac{\partial x^a}{\partial u^\alpha}\frac{\partial x^b}{\partial u^\beta}, \tag{2}$$

which is the 3-space metric inherited from the spacetime the matching surface is embedded in, be equal on both sides of the junction, as well as that the second fundamental form

$$\Upsilon_{\alpha\beta} = -n_{a||b}\frac{\partial x^a}{\partial u^\alpha}\frac{\partial x^b}{\partial u^\beta} = \left(\Gamma^c_{ab}n_c - n_{a|b}\right)\frac{\partial x^a}{\partial u^\alpha}\frac{\partial x^b}{\partial u^\beta}, \tag{3}$$

which describes the derivative of the unit normal vector to the hypersurface, be equal on both sides of the junction. The u^α co-ordinates are the co-ordinates of the 3-space of the hypersurface. The normal is given by

$$n_a = \frac{f_{|a}}{|g^{bc}f_{|b}f_{|c}|^{1/2}} \tag{4}$$

where f is a function of the co-ordinates such that it is zero on the junction. At first consideration it seems as though attempting to match an instant change from radiation domination to matter domination might violate the continuity across the junction: even if the scale factor can be matched at the junction to make the metric continuous, it appears that the rate of expansion would have to instantly change, making it difficult to match the metric connections. Thus, in §2 we will investigate the conditions that make it possible to match a radiation-dominated Einstein-de Sitter universe to a matter-dominated Einstein-de Sitter universe.

As an application of matching radiation-dominated universes to matter-dominated universes, the matching will be applied to all known asymptotically Einstein-de Sitter cosmological black hole spacetimes. These include M^cVittie's [4] isotropic black hole, which was generalized to the charged case by Gao and Zhang [5]; Vaidya's non-expanding Kerr-Schild black hole [6], which was generalized to the charged case by Patel and Trivedi [7]; and Thakurta's [8], Sultana and Dyer's [9], and M^cClure and Dyer's [10] expanding Schwarzschild black holes that were obtained via conformal transformation of the standard Schwarzschild, Eddington-Finkelstein, and isotropic black hole metrics respectively. It should be noted that while Vaidya, Patel and Trivedi, and Thakurta published metrics for cosmological Kerr black holes, there are no physical interpretations of the energy-momentum tensors that could give rise to these metrics, so solutions only exist for non-rotating cosmological black holes. In addition, most cosmological black hole solutions violate the energy conditions in some region of spacetime: only the M^cClure and Dyer cosmological black holes are physical throughout the spacetime. Also, despite the appearance of the metrics, the M^cVittie, Gao and Zhang, Vaidya, and Patel and Trivedi cosmological black hole spacetimes do not seem to have event horizons in the expected locations, so they may not actually represent black holes [10].

While cosmological black hole spacetimes look like the superposition of standard isolated black hole spacetimes in a cosmological background spacetime, the energy-momentum tensors can become complicated, so it would be difficult to deal with cosmological black holes in a two-fluid background universe. Yet at the same time, in order to devise solutions for primordial black holes, it necessitates having cosmological black hole solutions that can evolve to exist in different cosmological backgrounds. Although it is not very realistic of the manner the Universe would have evolved from being radiation dominated to matter dominated, being able to match radiation-dominated and matter-dominated cosmological black hole spacetimes would at least demonstrate that it is feasible to have cosmological black holes in evolving backgrounds. Thus, in §3 we will investigate the matching of cosmological black hole spacetimes to devise basic solutions for primordial black holes that evolve from being in radiation-dominated universes to matter-dominated universes.

2 Matching Einstein-de Sitter Universes

The Robertson-Walker metric for the Einstein-de Sitter universe is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + [R(t)]^2 \left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\right), \tag{5}$$

where $R(t) = (2H_o(t+C))^{1/2}$ in the case of a radiation-dominated universe and $R(t) = (3H_o(t+C)/2)^{2/3}$ in the case of a matter-dominated universe. The scale factor R is equal to 1 at the time t_o when the Hubble constant H_o is measured. The constant C (which arises from integrating the field equations) is generally zero if R is zero when t is zero. In the case of a universe that begins in one state and subsequently evolves according to another state, then C will be non-zero in the subsequent state to account for the difference in time it would have taken the universe to reach a given scale in the subsequent state from the time it actually took in the initial state.

Using the same (t, r, θ, ϕ) co-ordinates on both sides of the boundary, and with the bounding hypersurface a surface of constant t, then the same (r, θ, ϕ) co-ordinates will apply on the bounding hypersurface as well. Since g_{12}, g_{23}, and g_{13} are zero, the non-zero components of the first fundamental form will be given by

$$\Omega_{11} = g_{11} \tag{6a}$$

$$\Omega_{22} = g_{22} \tag{6b}$$

$$\Omega_{33} = g_{33}. \tag{6c}$$

Thus, the first fundamental form will match for the radiation-dominated and matter-dominated cases as long as R matches, which simply means the scale factor must be continuous across the junction. This matching can be achieved at any time t given a suitable choice of the constant C in the subsequent phase of the universe's expansion.

Looking at the second fundamental form, the unit normal can be calculated by considering that matching across a hypersurface of constant time implies f will be a function of t and will be zero on a hypersurface of constant time t_o for

$$f = t - t_o, \tag{7}$$

so the only non-zero derivative of f is $f_{|0} = 1$, which yields

$$n_a = \left(|g^{00}|^{-1/2}, 0, 0, 0 \right). \tag{8}$$

Thus, the second fundamental form reduces to

$$\Upsilon_{\alpha\beta} = \Gamma^{0}_{\alpha\beta} n_0. \tag{9}$$

The non-zero $\Gamma^{0}_{\alpha\beta}$ connections are

$$\Gamma^{0}_{11} = R\dot{R} \tag{10a}$$

$$\Gamma^{0}_{22} = r^2 R\dot{R} \tag{10b}$$

$$\Gamma^{0}_{33} = r^2 \sin^2\theta\, R\dot{R}, \tag{10c}$$

and n_0 will be the same for both the radiation-dominated and matter-dominated cases since g^{00} is the same. Thus, in order for the second fundamental form to be continuous across the boundary requires that both R and $\dot{R}$ be continuous across the junction.

In the case of a universe that begins radiation dominated and subsequently becomes matter dominated, then R will match with

$$(2H_o t)^{1/2} = \left(\frac{3}{2} H_o (t + C)\right)^{2/3}, \tag{11}$$

and $\dot{R}$ will match with

$$(2H_o)^{1/2} \frac{1}{2} t^{-1/2} = \left(\frac{3}{2} H_o\right)^{2/3} \frac{2}{3} (t + C)^{-1/3}, \tag{12}$$

which will both be satisfied for $H_o t = 1/2$ and $H_o C = 1/6$. Since $t = 1/(2H)$ for a radiation-dominated universe, then the matching must be performed at the time $t = t_o$ that H_o is measured and R is scaled to 1, but since H_o can be measured and R set to 1 at any choice of time for t_o, then it is possible to perform the matching at any time. In Figure 1 it can be seen that both the scale factor and its slope match at $H_o t = 1/2$ with $H_o C = 1/6$, allowing a radiation-dominated universe to smoothly match onto a matter-dominated universe. It should also be apparent that one could do the converse and match an initially matter-dominated universe onto a radiation-dominated universe at $H_o t = 2/3$ with $H_o C = -1/6$, which could be used to realistically model the situation of a universe initially filled with matter and antimatter that annihilates to leave radiation.

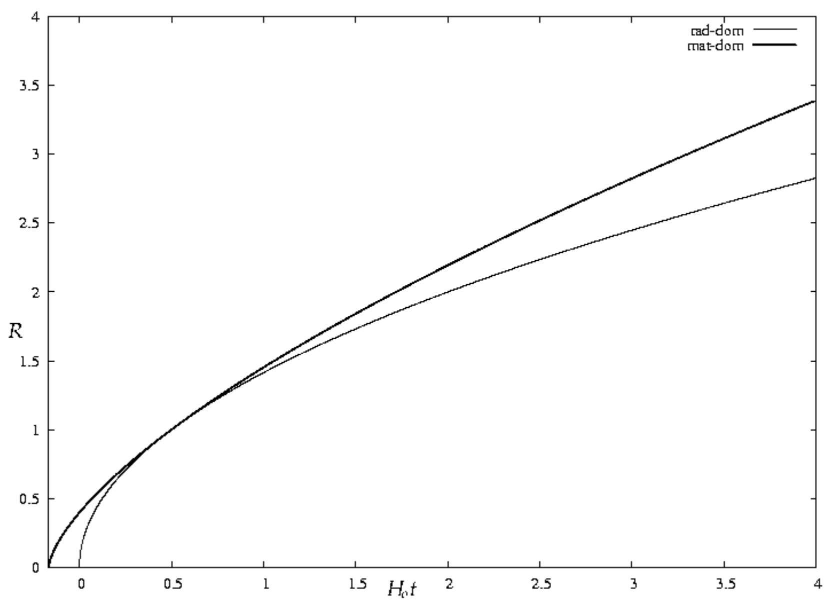

Figure 1: Evolution of the scale factor versus time, with the thin line representing the curve for a radiation-dominated universe, $R = (2H_o t)^{1/2}$, and the thick line representing the curve for a matter-dominated universe, $R = (3H_o(t + 1/6)/2)^{2/3}$. The scale factor and its slope both match when $R = 1$ at $H_o t = 1/2$.

In the case of a universe that begins matter dominated and subsequently becomes radiation dominated, R will match with

$$\left(\frac{3}{2}H_o t\right)^{2/3} = (2H_o(t+C))^{1/2}, \tag{13}$$

and $\dot{R}$ will match with

$$\left(\frac{3}{2}H_o\right)^{2/3}\frac{2}{3}t^{-1/3} = (2H_o)^{1/2}\frac{1}{2}(t+C)^{-1/2}, \tag{14}$$

which will indeed both be satisfied for $H_o t = 2/3$ and $H_o C = -1/6$. Since $t = 2/(3H)$ for a matter-dominated universe, then as before the matching must be performed at the time t_o that H_o is measured and R is set to 1, which arbitrarily may occur at any time.

3 Matching Cosmological Black Holes

The metric for an isotropic cosmological black hole that expands in an asymptotically Einstein-de Sitter universe [10] is

$$\mathrm{d}s^2 = -\left(\frac{1-m/(2r)}{1+m/(2r)}\right)^2 \mathrm{d}t^2 + R^2\left(1+\frac{m}{2r}\right)^4\left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right). \tag{15}$$

As with the plain Einstein-de Sitter universe, matching across a hypersurface of constant t and using the same (r,θ,ϕ) co-ordinates across the boundary, the first fundamental form will be continuous as long as $g_{\alpha\alpha}$ is continuous, and the second fundamental form will be continuous as long as $\Gamma^0_{\alpha\beta}n_0$ is continuous. Inspecting the metric, continuity of the metric just requires that the scale factor R is continuous. The non-zero $\Gamma^0_{\alpha\beta}$ connections are

$$\Gamma^0_{11} = \frac{(m+2r)^5}{16r^4(m-2r)}R\dot{R} \tag{16a}$$

$$\Gamma^0_{22} = \frac{(m+2r)^5}{16r^4(m-2r)}r^2R\dot{R} \tag{16b}$$

$$\Gamma^0_{33} = \frac{(m+2r)^5}{16r^4(m-2r)} r^2 \sin^2\theta \, R\dot{R}, \tag{16c}$$

and again n_0 is identical for both the radiation and matter cases. Thus, the conditions for the second fundamental form to be continuous and the Darmois conditions to be obeyed are that R and $\dot{R}$ are both continuous across the junction. Since the conditions only depend on the continuity of R and $\dot{R}$, then the matching is identical to the plain Einstein-de Sitter case. Once again the matching may be performed at any time as long as the scale factor is arbitrarily set to 1 at the time of the matching. Thus, it is possible to cut-and-paste together cosmological black holes in radiation-dominated and matter-dominated backgrounds.

Likewise, for all other known asymptotically Einstein-de Sitter cosmological black holes, it has been found that the only conditions for the matching are that R and $\dot{R}$ are both continuous, so the matching may be performed just as in the plain Einstein-de Sitter case. These cosmological black hole spacetimes include those of M^cVittie [4], Vaidya [6], Thakurta [8], and Sultana and Dyer [9]. This matching has also been found to hold true for the more general charged cases, which include Gao and Zhang's [5] charged McVittie spacetime, Patel and Trivedi's [7] charged Vaidya spacetime, and charged cases of the Thakurta, Sultana and Dyer, and McClure and Dyer black holes.

4 Conclusion

We have shown that it is possible to match a radiation-dominated Einstein-de Sitter universe directly to a matter-dominated Einstein-de Sitter universe across a hypersurface of constant time. Since it is possible to match a radiation-filled universe directly onto a matter-filled universe, this suggests that the composition of the Universe need not change continuously in time. Presumably, this would apply to the case of a more realistic two-fluid model as well, allowing some fraction of the photons or matter to instantaneously be converted. While such a transformation is not representative of the change between radiation domination and matter domination generally considered in cosmology with the CMB, it is representative of the conversions between radiation and matter that should have taken place in the early Universe.

In addition, we have shown that it is possible to match cosmological black hole spacetimes in radiation-dominated background universes and matter-dominated background universes. These matchings provide simplistic examples of primordial black holes that begin in radiation-dominated universes and evolve to be in matter-dominated universes. While such a direct matching is not very realistic, it does demonstrate that it is possible for cosmological black holes to exist in cosmological backgrounds of evolving composition.

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