Cosmological versions of Vaidya's radiating stellar exterior, an accelerating reference frame, and Kinnersley's photon rocket

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Abstract

The spacetimes for Vaidya's radiating stellar exterior and Kinnersley's photon rocket in a cosmological background are obtained by performing the same conformal transformation as is used to obtain the Robertson-Walker metric from Minkowski spacetime. In the case of the cosmological radiating stellar exterior, a two-fluid solution is found that obeys all of the energy conditions and consists of a null fluid and a perfect fluid that asymptotically falls off to the standard cosmological values for pressure and density at infinite radius. For the cosmological photon rocket, the massless case is first interpreted to obtain a solution for an accelerating cosmological reference frame, and then the general case is interpreted: in both cases, a two-fluid solution is found that consists of a null fluid and an imperfect fluid that possesses heat conduction and anisotropic stress. The imperfect fluid appears to contain an inhomogeneous dark energy component that acts to accelerate the matter through space via a pressure gradient, but this component has negative energy density on the trailing side of the rocket, meaning only the leading side of the rocket is guaranteed to satisfy the weak and dominant energy conditions. Unlike spacetimes that have rotation but no acceleration, the cosmological photon rocket can serve as an example of a spacetime that contradicts Mach's notion of acceleration, since an observer would see empirical evidence of acceleration even though the matter does not accelerate relative to the universe's background matter distribution.

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I. INTRODUCTION

A time-dependent conformal transformation of Minkowski space can yield the Robertson-Walker metric representing expanding matter-filled Friedmann universes. This is because the conformal transformation makes the spacetime dynamic and introduces mass-energy, which is consistent since a homogeneous matter content can cause an acceleration in the contraction of space backward-in-time toward the Big Bang, or equivalently cause a deceleration in the expansion of space forward-in-time, due to gravity. This same conformal transformation will be used in this paper on spacetimes related to Minkowski space (via a Kerr-Schild metric transformation [5]) with the goal of obtaining solutions in a cosmological background, ideally consisting of incoherent radiation or dust. These solutions should be less simplistic than the original solutions and less simplistic than homogeneous cosmological models, creating exact solutions of Einstein's field equations of greater sophistication.

Thakurta [1] performed a conformal transformation of the Schwarzschild metric to obtain a spacetime that locally looks like Schwarzschild and asymptotically looks like Robertson-Walker at infinite radius; however the energy-momentum tensor was not fully interpreted (Krasiński [2]), and it is known at least for the case of an asymptotically flat universe that it violates energy conditions within the event horizon (McClure and Dyer [3]). Sultana and Dyer [4] performed a conformal transformation of the Eddington-Finkelstein form (which is also a Kerr-Schild [5] form) of the Schwarzschild metric, yielding a different energy-momentum tensor that is unphysical within a radius that grows to infinity at infinite time; however, it can be shown that the white hole form yields a physical solution throughout the spacetime (as was calculated inadvertently by Sultana [6]). Thus, conformal transformations of spacetimes like the Eddington-Finkelstein white hole should be promising as potential cosmological solutions.

Vaidya [7] devised a metric for a radiating white hole by using the Schwarzschild solution and allowing the singularity mass to vary as a function of $u = t - r$ such that the white hole mass is radiated away on null surfaces. The metric was presented in Kerr-Schild form for the radiating Kerr white hole (Vaidya and Patel [8]). Kinnersley [9] devised a solution for a white hole that accelerates by radiating anisotropically, so it is the accelerated form of Vaidya's radiating stellar exterior. The photon rocket was presented in more detail and also extended to the charged case by Kinnersley and Walker [10].

In this paper solutions will be sought for the Kerr-Schild forms of Vaidya's radiating stellar exterior and Kinnersley's photon rocket in a cosmological background. Since the Kerr-Schild form of a white hole yields a solution in cosmological background [6], it can be expected that conformal transformations of the Kerr-Schild forms of the Vaidya and Kinnersley metrics may yield cosmological solutions. The calculations in this paper will be performed using the REDTEN package (Harper and Dyer [11]) and the computer algebra program REDUCE.

It should be noted that previously Vaidya and Patel [12] studied the Kerr metric in the Einstein static universe, which they assumed to be a mixture of a null fluid and perfect fluid, although they do not appear to have verified that this interpretation obeys the energy conditions. Obtaining solutions for cosmological Kerr black holes is problematic, since they must necessarily frame-drag the surrounding universe, so finding a solution for the exterior of a non-isolated Kerr black hole is somewhat like trying to find an exact solution for a Kerr interior. Thus, simply finding a solution of a spherically-symmetric radiating stellar exterior in an expanding universe should be more feasible and more realistic of our expanding Universe, so Kerr black holes will not be considered in this paper.

While it is expected that the conformal transformation should introduce a matter distribution similar to that of an Friedmann universe, an interesting problem is whether the photon rocket will accelerate with respect to the matter or whether it is possible for the entire matter distribution to accelerate along with the rocket. If the matter distribution does not accelerate with the rocket, it must avoid entering the event horizon of the photon rocket since it is a white hole. Also, the photon rocket has no gravitational radiation (Bonnor [13]) and the conformal transformation must preserve the causal structure such that there is no gravitational radiation despite the presence of the universe's background matter. Thus, finding a solution in which the background matter does not accelerate with the rocket seems more complicated than finding a solution in which it does.

If the matter distribution does accelerate with the rocket, it would be an unusual universe, but the existence of such a solution would be interesting as an example of a universe in which an object can have absolute acceleration without any acceleration relative to the universe's background matter distribution. This would be in contradiction to Mach's notion that acceleration is relative. While solutions with rotation such as the dust solution of Lanczos ([15]) have demonstrated that it is possible to have a universe with rotation, it should be noted that beyond Newtonian physics, rotation is not sufficient for the existence of acceleration. Since gravitational forces are geometrized in general relativity, matter that simply travels inertially along geodesics to undergo rotation actually has no force acting on it or accelerating it. Thus, to truly show that general relativity defies Mach's notion of acceleration, it is necessary to explore spacetimes in which there is actual acceleration of the matter, meaning spacetimes in which there are non-gravitational forces.

II. PRELIMINARIES

Kerr-Schild metrics [5] can be expressed as

$$\bar{g}_{ab} = g_{ab} + 2Hl_a l_b, \tag{1}$$

where g_{ab} is Minkowski spacetime (or in the generalized case, any seed metric), $2H$ is a scalar field, and l^a is a null vector field of the seed metric (and the transformed metric). The Schwarzschild metric can be expressed in Kerr-Schild form as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) + \frac{2m}{r}(\mathrm{d}t \pm \mathrm{d}r)^2, \tag{2}$$

with Minkowski spacetime as the seed metric, $2H = 2m/r$ as the scalar field, and $(1, \pm 1, 0, 0)$ as the null vector field. The plus or minus sign corresponds to the black hole or white hole case respectively (since it determines the sign of the $\mathrm{d}t\mathrm{d}r$ cross term in the metric so that the white hole case is the temporal reverse of the black hole case). This metric is related to the standard form of the Schwarzschild metric by a co-ordinate transformation and is equivalent to the Eddington-Finkelstein form of the Schwarzschild metric.

Vaidya's radiating stellar exterior [7] can be expressed in Kerr-Schild form as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) + \frac{2m(t,r)}{r}(\mathrm{d}t - \mathrm{d}r)^2, \tag{3}$$

where the white hole mass m varies across null surfaces as a function of $u = t - r$ such that

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{\mathrm{d}m}{\mathrm{d}r} = m' \tag{4}$$

as the white hole mass is radiated away. Looking at the Einstein tensor—with $G_{ab} = -\kappa T_{ab}$ (negative Einstein sign convention)—the only non-zero components are

$$G^0_0 = G^1_0 = -G^0_1 = -G^1_1 = -\frac{2m'}{r^2}, \tag{5}$$

which is interpreted as a null fluid. The energy-momentum tensor can be written as $T^{ab} = \tau l^a l^b$ with $l^a l_a = 0$ such that $G^a_a = 0$.

Kinnersley's photon rocket [9] with uniform acceleration a is

$$\begin{aligned} \mathrm{d}s^2 = &-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) + 2ar^2\sin\theta\,\mathrm{d}\theta(\mathrm{d}t - \mathrm{d}r) \\ &+ \left(a^2r^2\sin^2\theta + 2ar\cos\theta + \frac{2m(t,r)}{r}\right)(\mathrm{d}t - \mathrm{d}r)^2. \end{aligned} \tag{6}$$

In general the magnitude of the acceleration can vary and it can change direction, but the acceleration will be assumed to be uniform for the purposes of this paper. Looking at the Einstein tensor, the only non-zero components are

$$G^0_0 = G^1_0 = -G^0_1 = -G^1_1 = \frac{2}{r^2}(3am\cos\theta - m'), \tag{7}$$

which can be interpreted as a null fluid that radiates anisotropically to accelerate the white hole.

Sultana [6] obtained a solution for a white hole in Einstein-de Sitter dust (although it was referred to as a black hole solution) by performing a conformal transformation on the Eddington-Finkelstein form of the Schwarzschild metric with conformal factor $\Omega^2 = t^4$. Generalizing this cosmological white hole for a scale factor $R(t)$ for any flat Robertson-Walker metric, this spacetime can be expressed as

$$\mathrm{d}s^2 = [R(t)]^2\left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) + \frac{2m}{r}(\mathrm{d}t - \mathrm{d}r)^2\right), \tag{8}$$

where the time co-ordinate is related to the usual cosmological time co-ordinate—here designated as t_c—by a co-ordinate transformation $R(t)\mathrm{d}t = \mathrm{d}t_c$. For a flat radiation universe, the scale factor evolves compared with some time t_o at which $R = 1$ as $R(t_c) = (t_c/t_{c,o})^{1/2}$, which is equivalent to $R(t) = t/t_o$. For a flat dust universe the scale factor evolves as $R(t_c) = (t_c/t_{c,o})^{2/3}$, which is equivalent to $R(t) = (t/t_o)^2$.

The non-zero components of the Einstein tensor are

$$G^0_0 = \frac{3\dot{R}^2}{R^4}\left(1 + \frac{2m}{r}\right) + \frac{4\dot{R}m}{R^3r^2} \tag{9}$$

$$G^0_1 = \frac{2\dot{R}m}{R^3r^2} \tag{10}$$

$$G^1_0 = \left(\frac{2\dot{R}^2}{R^4} - \frac{\ddot{R}}{R^3}\right)\left(\frac{4m}{r}\right) - \frac{2\dot{R}m}{R^3r^2} \tag{11}$$

$$G^1_1 = -\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)\left(1 + \frac{2m}{r}\right) + \frac{8\dot{R}m}{R^3r^2} \quad (12)$$

$$G^2_2 = G^3_3 = -\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)\left(1 + \frac{2m}{r}\right). \quad (13)$$

The energy-momentum tensor can be interpreted as a two-fluid solution consisting of a null fluid and a perfect fluid. The null fluid is given by

$$\tau l^0 l_0 = \tau l^1 l_0 = -\tau l^0 l_1 = -\tau l^1 l_1 = \frac{2\dot{R}m}{\kappa R^3 r^2}. \quad (14)$$

The perfect fluid is given by

$$T^a_b = (\mu + p)u^a u_b + p g^a_b. \quad (15)$$

From spherical symmetry the angular components of the velocity field must be zero ($u^2 = u^3 = 0$), so the isotropic pressure is given from $G^2_2 = G^3_3 = -\kappa p$ as

$$p = \frac{1}{\kappa}\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)\left(1 + \frac{2m}{r}\right). \quad (16)$$

The energy density μ is determined from the trace of the Einstein tensor ($G^a_a = \kappa(\mu - 3p)$), which yields

$$\mu = \frac{3\dot{R}^2}{\kappa R^4}\left(1 + \frac{2m}{r}\right) + \frac{12\dot{R}m}{\kappa R^3 r^2}. \quad (17)$$

To satisfy the energy conditions (e.g. see Wald [14]), the weak energy condition requires that $\mu \geq 0$ and $\mu + p_i \geq 0$, the strong energy condition requires that $\mu + \Sigma p_i \geq 0$ and $\mu + p_i \geq 0$, and the dominant energy condition requires that $\mu \geq |p_i|$. For an expanding universe, $\dot{R}/R > 0$, the energy density is clearly positive everywhere for the cosmological white hole. For $R(t)$ that goes as t^x with $0 < x < 2$, the pressure is positive. Neglecting the last term of μ (which dominates at late times and is insignificant at small times compared with the pressure and the first part of the expression for μ) the pressure is $p >> \mu$ near $x = 0$, $p = \mu$ at $x = 1/2$, $p = \mu/3$ at $x = 1$, and $p = 0$ at $x = 2$. For $x > 2$ the pressure is negative, only asymptotically approaching $p \leq -\mu/3$. Since the last term of μ only serves to make μ larger for an expanding universe, then the pressure can only be less than or equal to the forementioned relations. Thus, clearly all the energy conditions are satisfied by the cosmological white hole at all times if the universe is expanding with $x > 1/2$ (which is in contrast to the corresponding cosmological black hole of Sultana and Dyer [4]).

III. A COSMOLOGICAL RADIATING STELLAR EXTERIOR

A conformal transformation can be performed on Vaidya's metric to obtain a metric that is asymptotically Roberston-Walker as r goes to infinity. For flat Robertson-Walker this yields

$$\mathrm{d}s^2 = [R(t)]^2 \left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 \right) + \frac{2m(t,r)}{r} (\mathrm{d}t - \mathrm{d}r)^2 \right), \tag{18}$$

where the scale factor $R(t)$ depends on t only. The non-zero Einstein tensor components of this metric are

$$G^0_0 = \frac{3\dot{R}^2}{R^4}\left(1 + \frac{2m}{r}\right) + \frac{2\dot{R}}{R^3 r}\left(\frac{2m}{r} - m'\right) - \frac{2m'}{R^2 r^2} \tag{19}$$

$$G^0_1 = \frac{2\dot{R}}{R^3 r}\left(\frac{m}{r} + m'\right) + \frac{2m'}{R^2 r^2} \tag{20}$$

$$G^1_0 = \left(\frac{2\dot{R}^2}{R^4} - \frac{\ddot{R}}{R^3}\right)\left(\frac{4m}{r}\right) - \frac{2\dot{R}}{R^3 r}\left(\frac{m}{r} + m'\right) - \frac{2m'}{R^2 r^2} \tag{21}$$

$$G^1_1 = -\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)\left(1 + \frac{2m}{r}\right) + \frac{2\dot{R}}{R^3 r}\left(\frac{4m}{r} + m'\right) + \frac{2m'}{R^2 r^2} \tag{22}$$

$$G^2_2 = G^3_3 = -\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)\left(1 + \frac{2m}{r}\right). \tag{23}$$

In the case where $m' = 0$, this reduces to the white-hole version of Sultana and Dyer's [4] cosmological black hole. The m' terms in G^0_0, G^0_1, G^1_0, and G^1_1 can be interpreted as an additional component of the null fluid due to the radiation of the point mass, analogous to Vaidya's radiating star. Thus, assuming a solution that consists of a superposition of a null fluid and a perfect fluid, then the null fluid component of the energy-momentum tensor is

$$\tau l^0 l_0 = \tau l^1 l_0 = -\tau l^0 l_1 = -\tau l^1 l_1 = \frac{2\dot{R}m}{\kappa R^3 r^2} + \frac{2m'\dot{R}}{\kappa R^3 r} + \frac{2m'}{\kappa R^2 r^2}. \tag{24}$$

Looking at the remaining terms of the Einstein tensor with the null fluid component $G^a_{b\,(nf)}$ subtracted reveals

$$G^0_0 - G^0_{0(nf)} = \frac{3\dot{R}^2}{R^4}\left(1 + \frac{2m}{r}\right) + \frac{6\dot{R}m}{R^3 r^2} \tag{25}$$

$$G^0_1 - G^0_{1(nf)} = 0 \tag{26}$$

$$G^1_0 - G^1_{0(nf)} = \left(\frac{2\dot{R}^2}{R^4} - \frac{\ddot{R}}{R^3}\right)\left(\frac{4m}{r}\right) \tag{27}$$

$$G_1^1 - G_{1(nf)}^1 = -\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)\left(1 + \frac{2m}{r}\right) + \frac{6\dot{R}m}{R^3r^2} \tag{28}$$

The remaining heat conduction component is due to the radial velocity field component u^1 of the perfect fluid, which leads to an effective heat conduction due to the flow of matter relative to the r co-ordinate. Spherical symmetry implies there can only be a radial component of the velocity field ($u^2 = u^3 = 0$), so $G_2^2 = G_3^3 = -\kappa p$ yields the isotropic pressure to be

$$p = \frac{1}{\kappa}\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)\left(1 + \frac{2m}{r}\right). \tag{29}$$

Since $G_a^a = 0$ for the null fluid component, then the energy density of the perfect fluid is determined from

$$G_a^a = \kappa(\mu - 3p) = \frac{6\dot{R}^2}{R^4}\left(1 + \frac{2m}{r}\right) + \frac{12\dot{R}m}{R^3r^2} \tag{30}$$

to be

$$\mu = \frac{3\dot{R}^2}{\kappa R^4}\left(1 + \frac{2m}{r}\right) + \frac{12\dot{R}m}{\kappa R^3r^2}. \tag{31}$$

The pressure and density of the perfect fluid are identical to that of the cosmological white hole in the previous section. Thus, just like that spacetime, this spacetime must represent a physical solution for a scale factor $R(t) = (t/t_o)^x$ with $x \geq 1/2$.

In the case of the scale factor for a radiation universe, $R(t) = t/t_o$, this yields

$$p = \frac{1}{\kappa R^2 t^2}\left(1 + \frac{2m}{r}\right) \tag{32}$$

and

$$\mu = \frac{3}{\kappa R^2 t^2}\left(1 + \frac{2m}{r}\right) + \frac{12m}{\kappa R^2 t r^2}, \tag{33}$$

so clearly μ and p are both positive and $p \leq \mu/3$, with the energy density and pressure both asymptotically approaching the standard radiation universe pressure and density as r goes to infinity. The equation of state is

$$p = \frac{\mu}{3} - \frac{4m}{\kappa R^2 t r^2}, \tag{34}$$

which is interpreted most simply if the energy density consists of a radiation component μ_r given by

$$\mu_r = \frac{3}{\kappa R^2 t^2}\left(1 + \frac{2m}{r}\right) \tag{35}$$

and a dust component μ_d given by

$$\mu_d = \frac{12m}{\kappa R^2 t r^2}, \tag{36}$$

so that the pressure corresponds to the radiation component and the dust component is pressureless.

In the case of the scale factor for a dust universe, $R(t) = (t/t_o)^2$, this yields $p = 0$ and

$$\mu = \frac{12}{\kappa R^2 t^2}\left(1 + \frac{2m}{r}\right) + \frac{24m}{\kappa R^2 t r^2}, \tag{37}$$

with the density again positive and falling off toward the standard dust universe density as r goes to infinity. Since $p = 0$, clearly this equation of state corresponds to pressureless dust.

It is assumed the terms of the Einstein tensor not corresponding to the null fluid can be represented by a perfect fluid with u^0 and u^1 components. If this is so, then

$$-\kappa[(\mu + p)u^0 u_0 + p] = \frac{3\dot{R}^2}{R^4}\left(1 + \frac{2m}{r}\right) + \frac{6\dot{R}m}{R^3 r^2} \tag{38}$$

$$-\kappa[(\mu + p)u^1 u_1 + p] = -\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)\left(1 + \frac{2m}{r}\right) + \frac{6\dot{R}m}{R^3 r^2} \tag{39}$$

with the T_0^1 component of the energy-momentum tensor merely being due to the u^1 component of the velocity field of the perfect fluid:

$$-\kappa[(\mu + p)u^1 u_0] = \left(\frac{2\dot{R}^2}{R^4} - \frac{\ddot{R}}{R^3}\right)\left(\frac{4m}{r}\right). \tag{40}$$

Using the interpreted expressions for μ (31) and p (29) implies that

$$u^0 u_0 = -\frac{\left(\frac{4\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)\left(1 + \frac{2m}{r}\right) + \frac{6\dot{R}m}{R^3 r^2}}{\left(\frac{4\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)\left(1 + \frac{2m}{r}\right) + \frac{12\dot{R}m}{R^3 r^2}} \tag{41}$$

and

$$u^1 u_1 = -\frac{\frac{6\dot{R}m}{R^3 r^2}}{\left(\frac{4\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)\left(1 + \frac{2m}{r}\right) + \frac{12\dot{R}m}{R^3 r^2}}, \tag{42}$$

verifying that

$$u^0 u_0 + u^1 u_1 = -1 = u^a u_a \tag{43}$$

as expected for a perfect fluid with a radial velocity field and no angular velocity field.

IV. AN ACCELERATING COSMOLOGICAL REFERENCE FRAME

Taking Kinnersley's photon rocket and setting the white hole mass to zero yields

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

$$+2ar^2\sin\theta\,\mathrm{d}\theta(\mathrm{d}t - \mathrm{d}r) + \left(a^2r^2\sin^2\theta + 2ar\cos\theta\right)(\mathrm{d}t - \mathrm{d}r)^2, \tag{44}$$

which is the metric for empty Minkowski space as seen from the reference frame of an accelerating observer at $r = 0$. Performing a conformal transformation on this accelerated version of Minkowski space yields

$$\mathrm{d}s^2 = [R(t)]^2\Big(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

$$+2ar^2\sin\theta\,\mathrm{d}\theta(\mathrm{d}t - \mathrm{d}r) + \left(a^2r^2\sin^2\theta + 2ar\cos\theta\right)(\mathrm{d}t - \mathrm{d}r)^2\Big). \tag{45}$$

Looking at the Einstein tensor, the non-zero components are

$$G^0_0 = \frac{3\dot{R}^2}{R^4}(1 + 2ar\cos\theta) + \frac{8\dot{R}}{R^3}(a\cos\theta) \tag{46}$$

$$G^0_1 = -\frac{2\dot{R}}{R^3}(a\cos\theta) \tag{47}$$

$$G^1_0 = \left(\frac{2\dot{R}^2}{R^4} - \frac{\ddot{R}}{R^3}\right)(4ar\cos\theta) + \frac{2\dot{R}}{R^3}(a\cos\theta) \tag{48}$$

$$G^0_2 = \frac{2\dot{R}}{R^3}(ar\sin\theta) \tag{49}$$

$$G^2_0 = -\left(\frac{2\dot{R}^2}{R^4} - \frac{\ddot{R}}{R^3}\right)(2a\sin\theta) - \frac{2\dot{R}}{R^3r}(a\sin\theta) \tag{50}$$

$$G^1_1 = -\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)(1 + 2ar\cos\theta) + \frac{4\dot{R}}{R^3}(a\cos\theta) \tag{51}$$

$$G^1_2 = \frac{2\dot{R}}{R^3}(ar\sin\theta) \tag{52}$$

$$G^2_1 = \frac{2\dot{R}}{R^3r}(a\sin\theta) \tag{53}$$

$$G^2_2 = G^3_3 = -\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)(1 + 2ar\cos\theta) + \frac{6\dot{R}}{R^3}(a\cos\theta). \tag{54}$$

The energy-momentum tensor appears to contain a null fluid with

$$\tau l^0 l_0 = \tau l^1 l_0 = -\tau l^0 l_1 = -\tau l^1 l_1 = -\frac{2\dot{R}}{\kappa R^3}(a\cos\theta), \tag{55}$$

where the null vector is $l^a = (1, -1, 0, 0)$. It is clear this does not fall off radially, so it is not consistent with radiation from $r = 0$. This suggests it is somehow being radiated by the fluid or is present at all times.

Looking at the remaining terms of the Einstein tensor with the null fluid component $G^a_{b\,(nf)}$ subtracted reveals

$$G^0_0 - G^0_{0(nf)} = \frac{3\dot{R}^2}{R^4}(1 + 2ar\cos\theta) + \frac{6\dot{R}}{R^3}(a\cos\theta) \tag{56}$$

$$G^0_1 - G^0_{1(nf)} = 0 \tag{57}$$

$$G^1_0 - G^1_{0(nf)} = \left(\frac{2\dot{R}^2}{R^4} - \frac{\ddot{R}}{R^3}\right)(4ar\cos\theta) \tag{58}$$

$$G^1_1 - G^1_{1(nf)} = G^2_2 = G^3_3 = -\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)(1 + 2ar\cos\theta) + \frac{6\dot{R}}{R^3}(a\cos\theta). \tag{59}$$

Due to axial symmetry, there must be no u^3 velocity field component, but since $G^1_1 - G^1_{1(nf)} = G^2_2 = G^3_3$, it then makes it impossible to interpret the remaining mass-energy component as an isotropic perfect fluid with non-zero u^1 and u^2. The energy-momentum tensor seems most logically interpreted with $u^1 = u^2 = u^3 = 0$, such that the fluid is an imperfect fluid that is comoving with the reference frame. An imperfect fluid can be written as

$$T^a_b = (\mu + p)u^a u_b + p g^a_b + q^a u_b + u^a q_b + \pi^a_b, \tag{60}$$

where q^a is the heat flow vector and π^a_b is the anisotropic stress. Thus, this appears to be an imperfect fluid with heat conduction components $q^1 u_0$, $u^0 q_2$, and $q^2 u_0$, and anisotropic stress components π^1_2 and π^2_1. The anisotropic stress must obey $\pi^a_b u^b = 0$, which with only u^0 non-zero means that $\pi^a_0 u^0 = 0$, which is satisfied as long as all $\pi^a_0 = 0$, so the T^a_0 components of the energy-momentum tensor contain a contribution due to q^a and not π^a_b. The heat conduction obeys $q^a u_a = 0$ and the anisotropic stress obeys $\pi^a_a = 0$, so these do not show up in the trace of the Einstein tensor.

Thus, it is possible to solve for the energy density and isotropic pressure via

$$G^a_a = \kappa(\mu - 3p) = \frac{6\ddot{R}}{R^3}(1 + 2ar\cos\theta) + \frac{24\dot{R}}{R^3}(a\cos\theta). \tag{61}$$

There must be no u^3 velocity field component. Assuming $\pi^3_3 = 0$, then $G^3_3 = -\kappa p$, which yields

$$p = \frac{1}{\kappa}\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)(1 + 2ar\cos\theta) - \frac{6\dot{R}}{\kappa R^3}(a\cos\theta), \tag{62}$$

and

$$\mu = \frac{3\dot{R}^2}{\kappa R^4}(1 + 2ar\cos\theta) + \frac{6\dot{R}}{\kappa R^3}(a\cos\theta). \tag{63}$$

For positive expansion $\dot{R}/R$ and positive $a\cos\theta$, the energy density is always positive. For negative $a\cos\theta$ the energy density can be negative. At small times $\dot{R}^2/R^4 >> \dot{R}/R^3$, so the sign of the energy density depends on $1 + 2ar\cos\theta$, where $r\cos\theta$ is essentially an axial co-ordinate z that specifies a plane in space; thus, the energy density is positive for $z > -1/(2a)$ at small times, so that in the limit as the acceleration goes to zero, the energy density is positive throughout the space. At late times $\dot{R}^2/R^4 << \dot{R}/R^3$, so the sign of the energy density depends on the sign of $a\cos\theta$, such that the energy density is only positive for $z > 0$.

At $\theta = \pi$, the pressure is positive for $R(t) = (t/t_0)^x$ with $0 < x \leq 2$, the pressure is zero for $x = 2$, and the pressure is negative for $x > 2$. At small times, the sign of the pressure reverses across a plane for $z = -1/(2a)$, but the magnitude of the pressure is no greater than $\mu/3$ as long as $x \geq 1$, so then the energy conditions are obeyed wherever the energy density is positive. At late times, the sign of the pressure depends on the sign of $-a\cos\theta$, such that the pressure is positive for $z < 0$ and negative for $z > 0$, and the pressure approaches $p = -\mu$, violating the strong energy condition.

In the case of a radiation universe scale factor, $R(t) = t/t_0$, the pressure is

$$p = \frac{1}{\kappa R^2 t^2}(1 + 2ar\cos\theta) - \frac{6}{\kappa R^2 t}(a\cos\theta) \tag{64}$$

and the energy density is

$$\mu = \frac{3}{\kappa R^2 t^2}(1 + 2ar\cos\theta) + \frac{6}{\kappa R^2 t}(a\cos\theta). \tag{65}$$

Thus, at small times the pressure is negative for $z < -1/(2a)$ and positive for $z > -1/(2a)$, while at large times the pressure becomes negative for $z > 0$ and positive for $z < 0$. The equation of state can be written as

$$p = \frac{\mu_r}{3} - \mu_\Lambda, \tag{66}$$

if the energy density is interpreted as having a radiation component μ_r given by

$$\mu_r = \frac{3}{\kappa R^2 t^2}(1 + 2ar\cos\theta) \tag{67}$$

and a component with Λ-type equation of state given by

$$\mu_\Lambda = \frac{6}{\kappa R^2 t}(a\cos\theta). \tag{68}$$

It should be noted that this Λ-type component cannot be interpreted geometrically as a cosmological constant since it is a function of θ and t. It could be interpreted as an inhomogeneous dark energy, although clearly it will yield a negative energy density for $z < 0$, so it violates the weak energy condition (and the dominant energy condition) and should be considered physically unrealistic for $z < 0$, unless we are willing to admit exotic matter as being physically plausible. While the dominant and weak energy conditions are obeyed for $z > 0$, the strong energy condition is violated for $z > 0$, but these facts are universally true of dark energy that has positive energy density, so it is in no way physically different that the strong energy condition is violated.

In the case of a dust universe scale factor, $R(t) = (t/t_0)^2$, the pressure is

$$p = -\frac{12}{\kappa R^2 t}(a\cos\theta) \tag{69}$$

and the energy density is

$$\mu = \frac{12}{\kappa R^2 t^2}(1 + 2ar\cos\theta) + \frac{12}{\kappa R^2 t}(a\cos\theta). \tag{70}$$

For all time the pressure is positive for $z < 0$ and negative for $z > 0$. There is only one term in the pressure gradient, so this appears to be what is causing the imperfect fluid to accelerate with the reference frame, although the heat conduction may also be involved as well. The equation of state can be written as

$$p = -\mu + \frac{12}{\kappa R^2 t^2}(1 + 2ar\cos\theta), \tag{71}$$

which is most easily explained if the energy density is assumed to have a pressureless dust component μ_d given by

$$\mu_d = \frac{12}{\kappa R^2 t^2}(1 + 2ar\cos\theta) \tag{72}$$

and, as above, an inhomogeneous dark energy component μ_Λ given by

$$\mu_\Lambda = \frac{12}{\kappa R^2 t}(a\cos\theta). \tag{73}$$

V. A COSMOLOGICAL PHOTON ROCKET

Performing a conformal transformation on Kinnersley's photon rocket yields

$$\mathrm{d}s^2 = [R(t)]^2 \Bigg(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right) + 2ar^2 \sin\theta\, \mathrm{d}\theta (\mathrm{d}t - \mathrm{d}r)$$

$$+ \left(a^2 r^2 \sin^2\theta + 2ar\cos\theta + \frac{2m(t,r)}{r} \right) (\mathrm{d}t - \mathrm{d}r)^2 \Bigg). \tag{74}$$

Looking at the Einstein tensor, the non-zero components are

$$G^0_0 = \frac{3\dot{R}^2}{R^4} \left(1 + 2ar\cos\theta + \frac{2m}{r}\right) + \frac{2\dot{R}}{R^3} \left(4a\cos\theta + \frac{2m}{r^2} - \frac{m'}{r}\right) + \frac{2}{R^2 r^2}(3am\cos\theta - m') \tag{75}$$

$$G^0_1 = \frac{2\dot{R}}{R^3} \left(-a\cos\theta + \frac{m}{r^2} + \frac{m'}{r}\right) - \frac{2}{R^2 r^2}(3am\cos\theta - m') \tag{76}$$

$$G^1_0 = \left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right) \left(4ar\cos\theta + \frac{4m}{r}\right) + \frac{2\dot{R}}{R^3} \left(a\cos\theta - \frac{m}{r^2} - \frac{m'}{r}\right)$$

$$+ \frac{2}{R^2 r^2}(3am\cos\theta - m') \tag{77}$$

$$G^0_2 = \frac{2\dot{R}}{R^3}(ar\sin\theta) \tag{78}$$

$$G^2_0 = -\left(\frac{2\dot{R}^2}{R^4} - \frac{\ddot{R}}{R^3}\right)(2a\sin\theta) - \frac{2\dot{R}}{R^3 r}\left(1 + \frac{3m}{r}\right) a\sin\theta \tag{79}$$

$$G^1_1 = -\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right) \left(1 + 2ar\cos\theta + \frac{2m}{r}\right) + \frac{2\dot{R}}{R^3} \left(2a\cos\theta + \frac{4m}{r^2} + \frac{m'}{r}\right)$$

$$- \frac{2}{R^2 r^2}(3am\cos\theta - m') \tag{80}$$

$$G^1_2 = \frac{2\dot{R}}{R^3}(ar\sin\theta) \tag{81}$$

$$G^2_1 = \frac{2\dot{R}}{R^3 r}\left(1 + \frac{3m}{r}\right) a\sin\theta \tag{82}$$

$$G^2_2 = G^3_3 = -\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right) \left(1 + 2ar\cos\theta + \frac{2m}{r}\right) + \frac{6\dot{R}}{R^3}(a\cos\theta). \tag{83}$$

The only terms that were not present in the $a = 0$ or $m = 0$ cases of the previous two sections are the $6am\cos\theta/(R^2 r)$ terms in G^0_0, G^0_1, G^1_0, and G^1_1, and the $6\dot{R}am\sin\theta/(R^2 r^2)$ terms in G^2_0 and G^2_1. The former are simply the anisotropic null fluid terms from Kinnersley's photon rocket scaled down by R^2, and the latter appear to be in the heat conduction and anisotropic stress terms of the imperfect fluid.

The null fluid is interpreted as

$$\tau l^0 l_0 = \tau l^1 l_0 = -\tau l^0 l_1 = -\tau l^1 l_1 = -\frac{2\dot{R}}{\kappa R^3}\left(a\cos\theta - \frac{m}{r^2} - \frac{m'}{r}\right) - \frac{2}{\kappa R^2 r^2}(3am\cos\theta - m'). \tag{84}$$

Looking at the remaining terms of the Einstein tensor with the null fluid component $G^a_{b\,(nf)}$ subtracted reveals

$$G^0_0 - G^0_{0(nf)} = \frac{3\dot{R}^2}{R^4}\left(1 + 2ar\cos\theta + \frac{2m}{r}\right) + \frac{6\dot{R}}{R^3}\left(a\cos\theta + \frac{m}{r^2}\right) \tag{85}$$

$$G^0_1 - G^0_{1(nf)} = 0 \tag{86}$$

$$G^1_0 - G^1_{0(nf)} = 4\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)\left(ar\cos\theta + \frac{m}{r}\right) \tag{87}$$

$$G^1_1 - G^1_{1(nf)} = -\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)\left(1 + 2ar\cos\theta + \frac{2m}{r}\right) + \frac{6\dot{R}}{R^3}\left(a\cos\theta + \frac{m}{r^2}\right). \tag{88}$$

Since u^3 must be zero due to axial symmetry, and $G^2_2 = G^3_3$, it appears that u^2 is once again zero as with the case of the accelerating cosmological reference frame, while $G^1_1 \neq G^3_3$ suggests that u^1 is once again non-zero as with the case of the cosmological radiating stellar exterior.

Again interpreting the remainder of the energy-momentum tensor as an imperfect fluid yields

$$G^a_a = \kappa(\mu - 3p) = \frac{6\ddot{R}}{R^3}\left(1 + 2ar\cos\theta + \frac{2m}{r}\right) + \frac{12\dot{R}}{R^3}\left(2a\cos\theta + \frac{m}{r^2}\right). \tag{89}$$

Since u^3 must be zero, the pressure is interpreted as $G^3_3 = -\kappa p$ so that

$$p = \frac{1}{\kappa}\left(\frac{\dot{R}^2}{R^4} - \frac{2\ddot{R}}{R^3}\right)\left(1 + 2ar\cos\theta + \frac{2m}{r}\right) - \frac{6\dot{R}}{\kappa R^3}(a\cos\theta), \tag{90}$$

which means that the energy density is given by

$$\mu = \frac{3\dot{R}^2}{\kappa R^4}\left(1 + 2ar\cos\theta + \frac{2m}{r}\right) + \frac{6\dot{R}}{\kappa R^3}\left(a\cos\theta + \frac{2m}{r^2}\right). \tag{91}$$

In the case of the accelerating cosmological reference frame of the previous section, the anisotropic stress obeys $\pi^a_0 u^0 = 0$, such that all π^a_0 have to be zero. In the case of the rocket with a radial velocity component u^1, the anisotropic stress must obey $\pi^a_0 u^0 + \pi^a_1 u^1 = 0$, so with $\pi_1^2 \neq 0$, it requires $\pi_0^2 \neq 0$, so the new terms $6\dot{R}am\sin\theta/(R^2r^2)$ in G_0^2 and G_1^2 appear to both be anisotropic stress terms in the imperfect fluid that result due to the mass of the rocket.

The additional terms in the energy density due to the mass of the rocket act to keep μ positive at small r, but part of μ is negative and causes it to violate the weak and dominant energy conditions for negative $a\cos\theta$. As with the $m = 0$ case, all energy conditions are satisfied for positive $a\cos\theta$, other than the strong energy condition that is violated at late times due to the dominance of the Λ-type component of the fluid.

VI. SUMMARY AND DISCUSSION

A physical solution of Einstein's field equations has been found for Vaidya's radiating stellar exterior in a cosmological background that is a two-fluid consisting of a null fluid and a cosmological perfect fluid. In the limit of infinite radius, the density and pressure are the same as the standard cosmological solutions.

It is interesting that cosmological white holes seem to more readily yield physical solutions than cosmological black holes do. Perhaps this is due to the tendency of black holes to oppose the expansion of the universe, since the expansion of space can cause outgoing null geodesics to expand instead of being trapped. For shrinking universes the sign of $\dot{R}/R$ is reversed, changing the energy density such that the cosmological white holes are unphysical and the cosmological black holes are physical. Thus, there is no physical preference for white hole solutions over black hole solutions, unless the universes of consideration are restricted to be expanding.

The spacetimes for an accelerating cosmological reference frame and Kinnersley's photon rocket in cosmological background have been interpreted as two-fluid models consisting of a null fluid and an imperfect fluid. On the leading side of the rocket, only the strong energy condition is violated at late times (as is always the case for a lambda-type equation of state with positive energy density), while on the trailing side the region satisfying the weak and dominant energy conditions is inversely related to the acceleration and the entire region on the trailing side of the rocket becomes unphysical at late times. Since acceleration is inversely related to the inertial mass $(\mu + p)$, it appears that in order to maintain the acceleration the energy density has to be negative on the trailing side of the rocket in order to offset the increasing pressure required to maintain the pressure gradient. If the energy density is interpreted as arising due to separate components, one of them being an inhomogeneous dark energy component, then this dark energy component has negative energy density on the trailing side of the rocket, so this interpretation requires the entire trailing side of the rocket to violate the weak energy condition and be unphysical for any finite acceleration.

While it may not be any more realistic to conceive of a universe that accelerates in bulk than one that undergoes bulk rotation, spacetimes for universes where the entire matter distribution undergoes acceleration serve to show that acceleration is absolute in general relativity and not simply relative. In the cosmological photon rocket spacetime, an observer will have empirical evidence that acceleration is taking place: without even observing the unusual mass-energy distribution of the universe and inferring that the matter is accelerating, an observer subject to the pressure gradient must directly "feel" the acceleration due to the unbalanced forces that act on the observer. This differs from spacetimes with rotation such as van Stockum dust (Lanczos [15]) and the Gödel universe [16] where the matter is simply moving inertially and there is no such empirically apparent pressure gradient or other force that acts to accelerate the matter.

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