Asymptotically Einstein-de Sitter cosmological black holes and the problem of energy conditions

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Abstract. It is shown that previous asymptotically Einstein-de Sitter cosmological black hole spacetimes violate the energy conditions in some region of spacetime. Thus, cosmological black hole spacetimes (for the Schwarzschild and Reissner-Nordström cases) are obtained that satisfy the energy conditions throughout spacetime. The solutions are obtained by performing conformal transformations on the isotropic forms of the isolated black hole metrics.

PACS numbers: 04.20.Jb, 04.70.Bw, 95.30.Sf, 97.60.Lf, 98.80.Jk

Submitted to: *Class. Quantum Grav.*

1. Introduction

Cosmological black hole spacetimes unite two relativistic extremes of small dense systems and large diffuse systems. They simultaneously provide examples of non-isolated black holes and inhomogeneous cosmological models. As black hole solutions, not only are they non-isolated, but they may be time-dependent, and need not be asymptotically flat (although for the purposes of this paper they will be). Since the presence of the black holes can modify the background density and pressure of the universe, and can also introduce heat conduction, they do not simply add a singularity into an otherwise homogeneous universe, so these spacetimes also make excellent inhomogeneous cosmological models.

Previous spacetimes for black holes embedded in FLRW universes are those of M^{c}Vittie (1933), generalized to the Reissner-Nordström case by Gao and Zhang (2004); Swiss cheese black holes (Einstein and Straus 1945); Vaidya's non-expanding cosmological Kerr spacetimes (1977), generalized to the Kerr-Newman case by Patel and Trivedi (1982); and Thakurta's (1981) and Sultana and Dyer's (2005) expanding cosmological Kerr and Schwarzschild black holes. Many of these spacetimes were outlined by Krasiński (1997).

The M^{c}Vittie spacetime was derived based on the assumptions that there is no flow of the matter and that the pressure is isotropic at every point in space. It is essentially the isotropic form of the Schwarzschild metric with the spatial part of the metric getting scaled by the expansion of the universe and the black hole mass getting scaled down by the scale factor. In the case of an asymptotically Einstein-de Sitter dust universe, M^{c}Vittie's spacetime yields a homogeneous mass density and a radially-varying pressure that falls off to zero only asymptotically. This spacetime was generalized to the Reissner-Nordström case by Gao and Zhang using the isotropic form of the Reissner-Nordström metric, which was first derived by Prasanna (1968).

Qualitatively, the Swiss cheese black holes can be constructed by cutting out spheres in an FLRW universe and contracting the dust spheres down into Schwarzschild black holes of the same mass. The matching of the Schwarzschild exterior onto the FLRW background can be shown to be valid using matching conditions such as the Darmois (1927) conditions to make sure that the metric at the matching surface is the same on both sides of the boundary and that the derivative of the unit normal vector to the hypersurface is the same from both sides of the boundary. More generally, the Lemaître-Tolman-Bondi models (Lemaître 1933, Tolman 1934, Bondi 1947) can be used to generate Swiss cheese black holes that are locally non-isolated, with the density smoothly increasing to that of the FLRW background universe. Swiss cheese black holes are useful for creating universes with multiple black holes, although they are also somewhat unrealistic in that they match the black holes onto a perfectly-FLRW background universe that is completely shielded from the influence of the black holes by the underdense regions surrounding them.

The Vaidya spacetime was derived by taking the Einstein static universe, performing a Kerr-Schild transformation (1965, also see Stephani *et al* 2003) to superimpose a Kerr black hole component onto the metric, and finally performing a conformal transformation (although not on the Kerr-Schild component of the metric) to change the black hole background from the Einstein static universe to a closed, expanding FLRW universe (effectively this is just like starting with a closed FLRW universe and doing a Kerr-Schild transformation). Patel and Trivedi extended Vaidya's Kerr case to the Kerr-Newman case.

Thakurta's spacetime was derived by performing a conformal transformation on the Boyer-Lindquist form of the isolated black hole metric to transform it into a Kerr black hole in an FLRW universe. Sultana and Dyer's spacetime was also derived via conformal transformation, but starting with the Eddington-Finkelstein form of the Schwarzschild metric (which is a Kerr-Schild form of the metric).

An FLRW universe is a conformal transformation of Minkowski space, and isolated black holes can be obtained via a Kerr-Schild transformation of Minkowski space. Thus, the Vaidya spacetime can be constructed from Minkowski space by performing a conformal transformation to get an FLRW universe, followed by a Kerr-Schild transformation to superimpose a black hole; the Sultana and Dyer black holes can be constructed from Minkowski space by performing a Kerr-Schild transformation to get an isolated black hole, followed by a conformal transformation to superimpose the black hole in an FLRW universe. The major difference between the Vaidya and the Sultana and Dyer spacetimes is that the Kerr-Schild component of the Vaidya spacetime does not expand with the universe, whereas the Kerr-Schild component of the Sultana and Dyer black holes gets transformed by the conformal transformation such that it becomes time-dependent and the black holes actually expand along with the universe.

It should be noted that while the Vaidya and Thakurta spacetimes are for Kerr black holes, the energy-momentum tensors can only be physically interpreted in the limit of zero angular momentum. Thus, the Kerr forms are not truly solutions of Einstein's field equations. Thakurta pointed out that the failure to obtain Kerr solutions is not surprising considering that no exact solution for a Kerr interior exists either. A non-isolated Kerr black hole would introduce rotation in the background universe via the Lense-Thirring effect (1918), which would require having a solution for a rotating source as with a Kerr interior solution. Interestingly, Nayak *et al* (2000) demonstrated that it is possible to match the Schwarzschild solution onto the Schwarzschild limit of the Vaidya cosmological black hole background, yet Cox (2003) demonstrated that it is not possible to match a Kerr black hole onto the Vaidya cosmological black hole background, suggesting that Vaidya's spacetime is not truly Kerr-like.

The Swiss cheese black holes satisfy the energy conditions: according to Einstein's field equations the energy-momentum tensor is determined locally by the spacetime, and both of the spacetimes that are cut and pasted together are known to satisfy the energy conditions. However, just because many of the asymptotically-FLRW black holes look like the superposition of a black hole spacetime with an FLRW universe, the field equations also demonstrate that they cannot be expected to be physically the same as superimposing the energy-momentum tensors for a black hole and an FLRW universe. Thus, it is important to make sure that the energy-momentum tensors can be interpreted and that they satisfy the weak, strong, and dominant energy conditions (e.g. see Wald 1984) to ensure that the spacetimes are physically possible and can indeed be deemed solutions of the field equations.

In this paper, the M$^{\rm c}$Vittie, Vaidya, Thakurta, and Sultana and Dyer spacetimes will be interpreted with particular attention to where the energy conditions are actually satisfied. The Vaidya spacetimes will be studied in the asymptotically-flat case (rather than the original closed form), and both the Thakurta and the Sultana and Dyer black holes will be generalized to the Reissner-Nordström case for the first time. In addition, new cosmological black hole spacetimes (derived like the Thakurta and the Sultana and Dyer spacetimes, but resembling the M$^{\rm c}$Vittie spacetime) will be derived that will be shown to satisfy the energy conditions throughout spacetime. The calculations will be performed using the computer algebra program REDUCE 3.5 (Hearn 1993) with the Redten 4.1e package (Harper and Dyer 1994).

2. M^{c}Vittie spacetimes

M$^{\rm c}$Vittie's metric (1933) in the Einstein-de Sitter case is

$$\mathrm{d}s^2 = -\left(\frac{1-\mu/(2r)}{1+\mu/(2r)}\right)^2 \mathrm{d}{t_c}^2 + [R(t_c)]^2\left(1+\frac{\mu}{2r}\right)^4\left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right), \tag{1}$$

where t_c is the cosmological time. The spatial part of the metric expands according to the scale factor R, and instead of the usual constant mass m that appears in the isotropic black hole metric, the mass $\mu = m/R$ varies and is scaled down by the expansion of the universe. In co-ordinates comoving with the universe's expansion the surface $r = m/(2R)$ shrinks according to the scale factor as the universe expands, but since the scale factor is scaling the r dimension according to R, then the net effect is that the spatial extent of this surface in non-comoving co-ordinates stays the same ($rR = m/2$).

It should be noted that the radial null curves satisfy

$$R\frac{\mathrm{d}r}{\mathrm{d}t_c} = \pm\frac{1-m/(2rR)}{(1+m/(2rR))^3}, \tag{2}$$

which suggests that while photons remain instantaneously motionless in r at $rR = m/2$ (since $\mathrm{d}r/\mathrm{d}t_c = 0$ there), they do not remain motionless in rR since it is not the case that $\mathrm{d}(rR)/\mathrm{d}t_c = 0$ there. It appears that as the radius $r = m/(2R)$ decreases with time, the photons will only momentarily be held at fixed r and will eventually be able to move outward as the surface $r = m/(2R)$ moves inward. Thus, it is not clear that the surface $rR = m/2$ acts as an event horizon, so the M$^{\rm c}$Vittie spacetime may not actually be a black hole, despite its resemblance to one.

Looking at the Einstein tensor (with $G_{ab} = -\kappa T_{ab}$), the only non-zero components are

$$G^0_0 = \frac{3\dot{R}^2}{R^2} \tag{3a}$$

$$G_1^1 = G_2^2 = G_3^3 = \frac{2(2rR+m)\ddot{R}/R + (2rR-5m)\dot{R}^2/R^2}{2rR-m}. \tag{3b}$$

Assuming a perfect fluid solution then

$$G_a^a = \kappa(\mu - 3p) = \frac{6(2rR+m)\ddot{R}/R + 8(rR-m)\dot{R}^2/R^2}{2rR-m}, \tag{4}$$

and assuming $G_1^1 = G_2^2 = G_3^3 = -\kappa p$ then

$$p = -\frac{2(2rR+m)\ddot{R}/R + (2rR-5m)\dot{R}^2/R^2}{\kappa(2rR-m)} \tag{5}$$

and

$$\mu = \frac{3\dot{R}^2}{\kappa R^2} \tag{6}$$

with no heat conduction. Thus, the energy density is spatially homogeneous, taking on the usual FLRW value, while the pressure is infinite at $r = m/(2R)$ and asymptotically approaches the usual FLRW pressure only as r goes to infinity.

In the case of a radiation-dominated background universe, $R = (2H_o t_c)^{1/2}$, the pressure and energy density will be given by

$$p = \frac{2rR+7m}{4\kappa {t_c}^2(2rR-m)} \tag{7}$$

and

$$\mu = \frac{3}{4\kappa {t_c}^2}. \tag{8}$$

Thus, it is apparent that inside $r = m/(2R)$, the pressure is negative, ranging from

$$p = -\frac{7}{4\kappa {t_c}^2} \tag{9}$$

as r approaches zero, to negative infinity as r approaches $r = m/(2R)$ from within. Outside $r = m/(2R)$, the pressure falls off from positive infinity at $r = m/(2R)$ to

$$p = \frac{1}{4\kappa {t_c}^2} \tag{10}$$

as r approaches infinity. Since the pressure is negative and greater in magnitude than the energy density everywhere inside $r = m/(2R)$, it violates all of the energy conditions there. Outside $r = m/(2R)$, the magnitude of the pressure is greater than that of the energy density for $r < 5m/(2R)$, so the dominant energy condition is also violated in that region.

In the case of a matter-dominated background universe, $R = (3H_o t_c/2)^{2/3}$, the pressure and energy density will be given by

$$p = \frac{8m}{3\kappa {t_c}^2(2rR-m)} \tag{11}$$

and

$$\mu = \frac{4}{3\kappa {t_c}^2}. \tag{12}$$

Thus, it is apparent that inside $r = m/(2R)$, the pressure is negative, ranging from

$$p = -\frac{8}{3\kappa {t_c}^2} \tag{13}$$

as r approaches zero, to negative infinity as r approaches $r = m/(2R)$ from within. Outside $r = m/(2R)$, the pressure falls off from positive infinity at $r = m/(2R)$, falling off as $1/r$ for large r and approaching zero as r goes to infinity. Since the pressure is negative and greater in magnitude than the energy density everywhere inside $r = m/(2R)$, it violates all of the energy conditions there. Outside $r = m/(2R)$, the magnitude of the pressure is greater than that of the energy density for $r < 3m/(2R)$, so the dominant energy condition is also violated in that region.

Gao and Zhang (2004) generalized M$^{\text{C}}$Vittie's spacetime to the Reissner-Nordström case. In the Einstein-de Sitter case, the metric is given by

$$\mathrm{d}s^2 = -\left(\frac{1 - m^2/(2rR)^2 + e^2/(2rR)^2}{(1 + m/(2rR))^2 - e^2/(2rR)^2}\right)^2 \mathrm{d}{t_c}^2 + \\ [R(t_c)]^2 \left(\left(1 + \frac{m}{2rR}\right)^2 - \left(\frac{e}{2rR}\right)^2\right)^2 (\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)), \tag{14}$$

so the mass m/R and charge e/R both vary and get scaled down by the expansion of the universe and the surface $r = \sqrt{m^2 - e^2}/(2R)$ shrinks in comoving co-ordinates, although they will all appear constant in an observer's non-comoving co-ordinates.

The non-zero Einstein tensor components are

$$G^0_0 = \frac{3\dot{R}^2}{R^2} + \frac{256e^2r^4R^4}{(e+m+2r)^4(e-m-2r)^4} \tag{15a}$$

$$G^1_1 = \frac{2(4r^2R^2 + 4rmR - e^2 + m^2)\ddot{R}/R + (4r^2R^2 - 8mrR + 5e^2 - 5m^2)\dot{R}^2/R^2}{4r^2R^2 + e^2 - m^2} + \\ \frac{256e^2r^4R^4}{(e+m+2r)^4(e-m-2r)^4} \tag{15b}$$

$$G^2_2 = G^3_3 = \frac{2(4r^2R^2 + 4rmR - e^2 + m^2)\ddot{R}/R + (4r^2R^2 - 8mrR + 5e^2 - 5m^2)\dot{R}^2/R^2}{4r^2R^2 + e^2 - m^2} - \\ \frac{256e^2r^4R^4}{(e+m+2r)^4(e-m-2r)^4}. \tag{15c}$$

Looking for a solution that consists of a perfect fluid plus electric field, the terms that go as R^4 clearly correspond to the electric field terms for the isolated isotropic Reissner-Nordström black hole

$$G^0_0 = G^1_1 = -G^2_2 = -G^3_3 = \frac{256e^2r^4}{(e+m+2r)^4(e-m-2r)^4}, \tag{16}$$

so the pressure and energy density are given by

$$p = -\frac{2(4r^2R^2 + 4rmR - e^2 + m^2)\ddot{R}/R + (4r^2R^2 - 8mrR + 5e^2 - 5m^2)\dot{R}^2/R^2}{\kappa(4r^2R^2 + e^2 - m^2)} \tag{17}$$

and

$$\mu = \frac{3\dot{R}^2}{\kappa R^2}. \tag{18}$$

In the radiation-dominated case the pressure is

$$p = \frac{4r^2R^2 + 16mrR - 7e^2 + 7m^2}{4\kappa(4r^2R^2 + e^2 - m^2)t_c^2}, \tag{19}$$

and the energy density is

$$\mu = \frac{3}{4\kappa t_c^2}, \tag{20}$$

which is spatially uniform, as with the M^{c}Vittie spacetime. The pressure becomes negative inside $r = \sqrt{m^2 - e^2}/(2R)$ where the denominator changes signs (the numerator can be considered positive since m should dominate e). Thus, inside $r = \sqrt{m^2 - e^2}/(2R)$ the pressure goes from

$$p = -\frac{7}{4\kappa {t_c}^2} \tag{21}$$

as r approaches zero, to negative infinity as r approaches $\sqrt{m^2 - e^2}/(2R)$ from within. Outside $r = \sqrt{m^2 - e^2}/(2R)$, the pressure falls off from positive infinity at $r = \sqrt{m^2 - e^2}/(2R)$ to

$$p = \frac{1}{4\kappa {t_c}^2} \tag{22}$$

as r approaches infinity. Since the pressure is negative and greater in magnitude than the energy density everywhere inside $r = \sqrt{m^2 - e^2}/(2R)$, it violates all of the energy conditions there, and outside $r = \sqrt{m^2 - e^2}/(2R)$, since the magnitude of the pressure will be greater than that of the energy density within some radius, the dominant energy condition is violated in that region.

In the matter-dominated case the pressure is

$$p = \frac{8(2mrR - e^2 + m^2)}{3\kappa(4r^2R^2 + e^2 - m^2)t_c^2}, \tag{23}$$

and the energy density is

$$\mu = \frac{4}{3\kappa {t_c}^2}, \tag{24}$$

so the energy density is again uniform, cf. (12). The pressure varies from

$$p = -\frac{8}{3\kappa {t_c}^2} \tag{25}$$

as r approaches zero, to negative infinity as r approaches $\sqrt{m^2 - e^2}/(2R)$ from within, and from positive infinity as r approaches $\sqrt{m^2 - e^2}/(2R)$ from outside to zero as r approaches infinity. Similar to the uncharged M^{c}Vittie spacetime, all the energy conditions are violated inside $r = \sqrt{m^2 - e^2}/(2R)$, and the dominant energy condition is violated within some radius beyond $\sqrt{m^2 - e^2}/(2R)$.

3. Vaidya spacetimes

A Vaidya metric (1977) in the case of an asymptotically-flat universe

$$\mathrm{d}s^2 = R^2\left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2)\right) + \frac{2m}{r}(\mathrm{d}t + \mathrm{d}r)^2 \tag{26}$$

(where t is just a co-ordinate time, not the cosmological time) represents an Einstein-de Sitter-like universe with a Schwarzschild-like black hole that does not expand with the rest of the universe.

It should be noted that Vaidya claims the event horizon is at $g^{rr} = 0$, which implies the event horizon is at $r = 2m/R^2$. However, the radial null curves satisfy

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{R^2 - 2m/r}{R^2 + 2m/r}, \tag{27}$$

which suggests that while photons are instantaneously motionless in r at $r = 2m/R^2$, the radius of the surface where $\mathrm{d}r/\mathrm{d}t = 0$ is shrinking to smaller r with time, suggesting outgoing photons should only momentarily be held at fixed r and not remain trapped as $r = 2m/R^2$ shrinks inward with time. By calculating the normal vector to the surface given by $r = 2m/R^2$, it can be verified that it is null only for $4m\dot{R} = R^3$, which can be made null at a given time via the arbitrary magnitude of the scale factor, but for a scale factor that varies with time as for a radiation or dust universe, this cannot remain null for all time. Thus, while this surface implies the inhomogeneity appears to decrease in mass and shrink in radius in time (in both comoving co-ordinates and an observer's non-comoving co-ordinates), it does not appear to be an event horizon, so it is unclear whether the Vaidya spacetime actually represents a black hole.

The metric has non-zero Einstein tensor components

$$G^0_0 = \frac{3\dot{R}^2}{R^4} + \frac{2m\dot{R}^2}{rR^6} - \frac{4m\dot{R}}{r^2R^5} \tag{28a}$$

$$G^0_1 = -\frac{2m\dot{R}}{r^2R^5} \tag{28b}$$

$$G^1_0 = \frac{4m\ddot{R}}{rR^5} - \frac{8m\dot{R}^2}{rR^6} + \frac{2m\dot{R}}{r^2R^5} \tag{28c}$$

$$G^1_1 = \frac{2\ddot{R}}{R^3} + \frac{4m\ddot{R}}{rR^5} - \frac{\dot{R}^2}{R^4} - \frac{6m\dot{R}^2}{rR^6} \tag{28d}$$

$$G^2_2 = G^3_3 = \frac{2\ddot{R}}{R^3} + \frac{2m\ddot{R}}{rR^5} - \frac{\dot{R}^2}{R^4} - \frac{4m\dot{R}^2}{rR^6}. \tag{28e}$$

Looking for a solution that consists of a perfect fluid plus heat conduction, then since $T^a_a = 0$ for the heat conduction component,

$$G^a_a = \kappa(\mu - 3p) = -\frac{4m\dot{R}}{r^2R^5} + \frac{6\ddot{R}}{R^3} + \frac{8m\ddot{R}}{rR^5} - \frac{12m\dot{R}^2}{rR^6}. \tag{29}$$

Assuming $G^2_2 = G^3_3 = -\kappa p$, then

$$p = -\frac{2\ddot{R}}{\kappa R^3} - \frac{2m\ddot{R}}{\kappa rR^5} + \frac{\dot{R}^2}{\kappa R^4} + \frac{4m\dot{R}^2}{\kappa rR^6} \tag{30}$$

and

$$\mu = \frac{2m\ddot{R}}{\kappa r^2 R^5} + \frac{3\dot{R}^2}{\kappa R^4} - \frac{4m\dot{R}}{\kappa r^2 R^5}. \tag{31}$$

In the radiation-dominated case, $R = H_o t$, the pressure is

$$p = \frac{1}{\kappa R^2 t^2} + \frac{4m}{\kappa r R^4 t^2} \tag{32}$$

and the energy density is

$$\mu = \frac{3}{\kappa R^2 t^2} - \frac{4m}{\kappa r^2 R^4 t}. \tag{33}$$

The energy density will become negative and the weak energy condition will be violated for

$$4mt > 3r^2 R^2. \tag{34}$$

Since R^2 grows as t^2, this is a problem for small r and small t so that it becomes problematic for larger values of r as the Big Bang is approached. However, since R can arbitrarily be scaled up at any given time simply by setting the scale factor to 1 earlier in time, it suggests the invalid region of spacetime can be made arbitrarily small, although R cannot be set to 1 immediately at $t = 0$ so there will always be some region of spacetime that will violate the weak energy condition. Since the pressure is non-zero, then the requirement that the pressure be smaller in magnitude than the energy density will cause the dominant energy condition to be violated even before the energy density goes negative and the weak energy condition is violated.

In the matter-dominated case, $R = (H_o t/2)^2$, the pressure is

$$p = \frac{12m}{\kappa r R^4 t^2} \tag{35}$$

and the energy density is

$$\mu = \frac{12}{\kappa R^2 t^2} + \frac{4m}{\kappa r R^4 t^2} - \frac{8m}{\kappa r^2 R^4 t}. \tag{36}$$

The energy density will become negative and the weak energy condition will be violated for

$$2mt > 3r^2 R^2 + 4mr. \tag{37}$$

As before, by setting R to 1 at earlier times, a smaller region of spacetime will violate the energy conditions.

In the Reissner-Nordström case, which is the $a = 0$ limit of the Patel and Trivedi (1982) spacetime,

$$\mathrm{d}s^2 = R^2 \left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\right) + \left(\frac{2m}{r} - \frac{e^2}{r^2}\right)(\mathrm{d}t + \mathrm{d}r)^2. \tag{38}$$

The metric has non-zero Einstein tensor components

$$G^0_0 = \frac{3\dot{R}^2}{R^4} - \frac{e^2\dot{R}^2}{r^2R^6} + \frac{2m\dot{R}^2}{rR^6} - \frac{4m\dot{R}}{r^2R^5} + \frac{e^2}{r^4R^4} \tag{39a}$$

$$G^0_1 = -\frac{2m\dot{R}}{r^2R^5} \tag{39b}$$

$$G^1_0 = -\frac{2e^2\ddot{R}}{r^2R^5} + \frac{4m\ddot{R}}{rR^5} + \frac{4e^2\dot{R}^2}{r^2R^6} - \frac{8m\dot{R}^2}{rR^6} + \frac{2m\dot{R}}{r^2R^5} \tag{39c}$$

$$G^1_1 = \frac{2\ddot{R}}{R^3} - \frac{2e^2\ddot{R}}{r^2R^5} + \frac{4m\ddot{R}}{rR^5} - \frac{\dot{R}^2}{R^4} + \frac{3e^2\dot{R}^2}{r^2R^6} - \frac{6m\dot{R}^2}{rR^6} + \frac{e^2}{r^4R^4} \tag{39d}$$

$$G^2_2 = G^3_3 = \frac{2\ddot{R}}{R^3} - \frac{e^2\ddot{R}}{r^2R^5} + \frac{2m\ddot{R}}{rR^5} - \frac{\dot{R}^2}{R^4} + \frac{2e^2\dot{R}^2}{r^2R^6} - \frac{4m\dot{R}^2}{rR^6} - \frac{e^2}{r^4R^4}. \tag{39e}$$

The $e^2/(r^4R^4)$ terms correspond to the electric field terms for the isolated Reissner-Nordström metric

$$G^0_0 = G^1_1 = -G^2_2 = -G^3_3 = \frac{e^2}{r^4}, \tag{40}$$

while the terms involving e^2/r^2 appear to modify the heat conduction, energy density, and pressure. Since $T^a_a = 0$ for the electric and heat conduction components, this interpretation yields

$$G^a_a = \kappa(\mu - p) = -\frac{4m\dot{R}}{r^2R^5} - \frac{4e^2\ddot{R}}{r^2R^5} + \frac{6\ddot{R}}{R^3} + \frac{8m\ddot{R}}{rR^5} + \frac{6e^2\dot{R}^2}{r^2R^6} - \frac{12m\dot{R}^2}{rR^6}, \tag{41}$$

which with $G^2_2 = G^3_3 = -\kappa p$, then

$$p = -\frac{2\ddot{R}}{\kappa R^3} + \frac{e^2\ddot{R}}{\kappa r^2R^5} - \frac{2m\ddot{R}}{\kappa rR^5} + \frac{\dot{R}^2}{\kappa R^4} - \frac{2e^2\dot{R}^2}{\kappa r^2R^6} + \frac{4m\dot{R}^2}{\kappa rR^6} \tag{42}$$

and

$$\mu = -\frac{4m\dot{R}}{\kappa r^2R^5} - \frac{e^2\ddot{R}}{\kappa r^2R^5} + \frac{2m\ddot{R}}{\kappa rR^5} + \frac{3\dot{R}^2}{\kappa R^4}. \tag{43}$$

In the radiation-dominated case

$$p = \frac{1}{\kappa R^2t^2} + \frac{4m}{\kappa rR^4t^2} - \frac{2e^2}{\kappa r^2R^4t^2} \tag{44}$$

and

$$\mu = \frac{3}{\kappa R^2t^2} - \frac{4m}{\kappa r^2R^4t}, \tag{45}$$

and in the matter-dominated case

$$p = \frac{12m}{\kappa rR^4t^2} - \frac{6e^2}{\kappa r^2R^4t^2} \tag{46}$$

and

$$\mu = \frac{12}{\kappa R^2t^2} - \frac{8m}{\kappa r^2R^4t} + \frac{4m}{\kappa rR^4t^2} - \frac{2e^2}{\kappa r^2R^4t^2}. \tag{47}$$

Thus, in the radiation-dominated case, the energy density is unaffected by the charge so it should make no difference as far as violating the weak energy condition, while in the matter-dominated case it decreases the energy density so that the weak energy condition will be violated more easily. However, the largest charge-to-mass ratio that exists is that of an electron, and $2m/r$ dominates e^2/r^2 for $r < 1.5 \times 10^{-15}$ m, which is the scale where the strong force comes into play, so the effect of the charge should be negligible except in instances where different physics should apply anyway.

4. Thakurta black holes

The Thakurta metric (1981) in the zero angular momentum limit

$$\mathrm{d}s^2 = R^2\left(-\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right) \tag{48}$$

represents an Einstein-de Sitter-like universe with a Schwarzschild-like black hole. The event horizon is at $r = 2m$, since this is just a conformal transformation of the Schwarzschild metric. Since the r co-ordinate gets scaled by the scale factor, this means the black hole expands with the universe. In co-ordinates comoving with the expansion, the effective mass of the black hole will remain constant, while in an observer's non-comoving co-ordinates, the effective mass of the black hole will appear to increase.

The metric has non-zero Einstein tensor components

$$G^0_0 = -\frac{3r\dot{R}^2}{(2m-r)R^4} \tag{49a}$$

$$G^0_1 = -\frac{2m\dot{R}}{(2m-r)^2R^3} \tag{49b}$$

$$G^1_0 = -\frac{2m\dot{R}}{r^2R^3} \tag{49c}$$

$$G^1_1 = G^2_2 = G^3_3 = -\frac{2r\ddot{R}}{(2m-r)R^3} + \frac{r\dot{R}^2}{(2m-r)R^4}. \tag{49d}$$

Looking for a solution that consists of a perfect fluid plus heat conduction yields

$$G^a_a = \kappa(\mu - 3p) = -\frac{6r\ddot{R}}{(2m-r)R^3}, \tag{50}$$

which with

$$p = \frac{2r\ddot{R}}{\kappa(2m-r)R^3} - \frac{r\dot{R}^2}{\kappa(2m-r)R^4} \tag{51}$$

yields

$$\mu = -\frac{3r\dot{R}^2}{\kappa(2m-r)R^4}. \tag{52}$$

In the radiation-dominated case

$$p = -\frac{r}{\kappa(2m-r)R^2t^2} \tag{53}$$

and

$$\mu = -\frac{3r}{\kappa(2m-r)R^2t^2}, \tag{54}$$

so within the event horizon the energy density and pressure are both negative and violate all the energy conditions. Outside the event horizon, the energy density and pressure both reverse signs so that it becomes a valid solution, and the pressure is simply one third of the energy density as expected for a radiation-dominated universe. In the matter-dominated case the pressure is zero as expected and

$$\mu = -\frac{12r}{\kappa(2m - r)R^2t^2}, \tag{55}$$

so the energy conditions are violated inside the event horizon as with the radiation-dominated case.

Extending this to the Reissner-Nordström case then

$$\mathrm{d}s^2 = R^2\left(-\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right). \tag{56}$$

The metric has non-zero Einstein tensor components

$$G^0_0 = \frac{3r^2\dot{R}^2}{(e^2 - 2mr + r^2)R^4} + \frac{e^2}{r^4R^2} \tag{57a}$$

$$G^0_1 = \frac{2mr^2\dot{R}}{(e^2 - 2mr + r^2)^2R^3} - \frac{2e^2r\dot{R}}{(e^2 - 2mr + r^2)^2R^3} \tag{57b}$$

$$G^1_0 = -\frac{2m\dot{R}}{r^2R^3} + \frac{2e^2\dot{R}}{r^3R^3} \tag{57c}$$

$$G^1_1 = \frac{2r^2\ddot{R}}{(e^2 - 2mr + r^2)R^3} - \frac{r^2\dot{R}^2}{(e^2 - 2mr + r^2)R^4} + \frac{e^2}{r^4R^2} \tag{57d}$$

$$G^2_2 = G^3_3 = \frac{2r^2\ddot{R}}{(e^2 - 2mr + r^2)R^3} - \frac{r^2\dot{R}^2}{(e^2 - 2mr + r^2)R^4} - \frac{e^2}{r^4R^2}. \tag{57e}$$

Looking for a solution that consists of a perfect fluid plus heat conduction plus electric field yields

$$G^a_a = \kappa(\mu - 3p) = \frac{6r^2\ddot{R}}{(e^2 - 2mr + r^2)R^3}, \tag{58}$$

which with

$$p = -\frac{2r^2\ddot{R}}{\kappa(e^2 - 2mr + r^2)R^3} + \frac{r^2\dot{R}^2}{\kappa(e^2 - 2mr + r^2)R^4} \tag{59}$$

yields

$$\mu = \frac{3r^2\dot{R}^2}{\kappa(e^2 - 2mr + r^2)R^4}. \tag{60}$$

In the radiation-dominated case

$$p = \frac{r^2}{\kappa(e^2 - 2mr + r^2)R^2t^2} \tag{61}$$

and

$$\mu = \frac{3r^2}{\kappa(e^2 - 2mr + r^2)R^2t^2}, \tag{62}$$

so within the event horizon the energy density and pressure are both negative and violate all the energy conditions. Outside the event horizon, the energy density and pressure both reverse signs so that it becomes a valid solution, and the pressure is simply one third of the energy density as expected for a radiation-dominated universe. In the matter-dominated case the pressure is zero as expected and

$$\mu = \frac{12r^2}{\kappa(e^2 - 2mr + r^2)R^2t^2}, \tag{63}$$

so the energy conditions are violated inside the event horizon as with the radiation-dominated case.

5. Sultana and Dyer black holes

The Sultana and Dyer (2005) black hole metric

$$\mathrm{d}s^2 = R^2\left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + \frac{2m}{r}(\mathrm{d}t + \mathrm{d}r)^2\right) \tag{64}$$

represents an Einstein-de Sitter-like universe with a Schwarzschild-like black hole. The event horizon is at $r = 2m$, since this is just a conformal transformation of the Eddington-Finkelstein form of the Schwarzschild metric. The r co-ordinate gets scaled by the scale factor, so the black hole expands with the universe. In co-ordinates comoving with the expansion, the effective mass of the black hole will remain constant, while in non-comoving co-ordinates, the effective mass of the black hole will appear to increase.

The metric has non-zero Einstein tensor components

$$G^0_0 = \frac{6m\dot{R}^2}{rR^4} + \frac{3\dot{R}^2}{R^4} - \frac{4m\dot{R}}{r^2R^3} \tag{65a}$$

$$G^0_1 = \frac{2m\dot{R}}{r^2R^3} \tag{65b}$$

$$G^1_0 = \frac{4m\ddot{R}}{rR^3} - \frac{8m\dot{R}^2}{rR^4} - \frac{2m\dot{R}}{r^2R^3} \tag{65c}$$

$$G^1_1 = \frac{4m\ddot{R}}{rR^3} + \frac{2\ddot{R}}{R^3} - \frac{2m\dot{R}^2}{rR^4} - \frac{\dot{R}^2}{R^4} - \frac{8m\dot{R}}{r^2R^3} \tag{65d}$$

$$G^2_2 = G^3_3 = \frac{4m\ddot{R}}{rR^3} + \frac{2\ddot{R}}{R^3} - \frac{2m\dot{R}^2}{rR^4} - \frac{\dot{R}^2}{R^4}. \tag{65e}$$

Looking for a solution that consists of a perfect fluid plus heat conduction yields

$$G^a_a = \kappa(\mu - 3p) = \frac{12m\ddot{R}}{rR^3} - \frac{12m\dot{R}}{r^2R^3} + \frac{6\ddot{R}}{R^3}, \tag{66}$$

which with $G^2_2 = G^3_3 = -\kappa p$ yields

$$p = -\frac{4m\ddot{R}}{\kappa rR^3} - \frac{2\ddot{R}}{\kappa R^3} + \frac{2m\dot{R}^2}{\kappa rR^4} + \frac{\dot{R}^2}{\kappa R^4} \tag{67}$$

and

$$\mu = -\frac{12m\dot{R}}{\kappa r^2R^3} + \frac{6m\dot{R}^2}{\kappa rR^4} + \frac{3\dot{R}^2}{\kappa R^4}. \tag{68}$$

In the radiation-dominated case

$$p = \frac{1}{\kappa R^2 t^2} + \frac{2m}{\kappa r R^2 t^2} \tag{69}$$

and

$$\mu = \frac{3}{\kappa R^2 t^2} + \frac{6m}{\kappa r R^2 t^2} - \frac{12m}{\kappa r^2 R^2 t}. \tag{70}$$

The weak energy condition will be violated for

$$4mt > 2mr + r^2, \tag{71}$$

and the dominant energy condition will be violated for

$$6mt > 2mr + r^2, \tag{72}$$

which is problematic for large times or small radii. In the matter-dominated case $p = 0$ and

$$\mu = \frac{12}{\kappa R^2 t^2} + \frac{24m}{\kappa r R^2 t^2} - \frac{24m}{\kappa r^2 R^2 t}, \tag{73}$$

so the weak energy condition is violated for

$$2mt > 2mr + r^2, \tag{74}$$

which once again is a problem for large times and small radii. Thus, unlike the Vaidya black holes, the Sultana black holes become problematic for larger values of r at later stages of time.

Extending this to the Reissner-Nordström case then

$$\mathrm{d}s^2 = R^2 \left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2) + \left(\frac{2m}{r} - \frac{e^2}{r^2} \right) (\mathrm{d}t + \mathrm{d}r)^2 \right). \tag{75}$$

The metric has non-zero Einstein tensor components

$$G^0_0 = -\frac{3e^2\dot{R}^2}{r^2R^4} + \frac{6m\dot{R}^2}{rR^4} + \frac{3\dot{R}^2}{R^4} - \frac{4m\dot{R}}{r^2R^3} + \frac{e^2}{r^4R^2} \tag{76a}$$

$$G^0_1 = -\frac{2e^2\dot{R}}{r^3R^3} + \frac{2m\dot{R}}{r^2R^3} \tag{76b}$$

$$G^1_0 = -\frac{2e^2\ddot{R}}{r^2R^3} + \frac{4m\ddot{R}}{rR^3} + \frac{4e^2\dot{R}^2}{r^2R^4} - \frac{8m\dot{R}^2}{rR^4} + \frac{2e^2\dot{R}}{r^3R^3} - \frac{2m\dot{R}}{r^2R^3} \tag{76c}$$

$$G^1_1 = -\frac{2e^2\ddot{R}}{r^2R^3} + \frac{4m\ddot{R}}{rR^3} + \frac{2\ddot{R}}{R^3} + \frac{e^2\dot{R}^2}{r^2R^4} - \frac{2m\dot{R}^2}{rR^4} - \frac{\dot{R}^2}{R^4} + \frac{4e^2\dot{R}}{r^3R^3} - \frac{8m\dot{R}}{r^2R^3} + \frac{e^2}{r^4R^2} \tag{76d}$$

$$G^2_2 = G^3_3 = -\frac{2e^2\ddot{R}}{r^2R^3} + \frac{4m\ddot{R}}{rR^3} + \frac{2\ddot{R}}{R^3} + \frac{e^2\dot{R}^2}{r^2R^4} - \frac{2m\dot{R}^2}{rR^4} - \frac{\dot{R}^2}{R^4} - \frac{2e^2\dot{R}}{r^3R^3} - \frac{e^2}{r^4R^2}. \tag{76e}$$

The $e^2/(r^4R^2)$ terms correspond to the usual Reissner-Nordström terms for the electric field (40), and the other terms involving charge appear to modify the heat conduction, energy density, and pressure. The $4e^2\dot{R}/(r^3R^3)$ term in G^1_1 cancels the $-2e^2\dot{R}/(r^3R^3)$ terms in $G^2_2 = G^3_3$, suggesting these terms modify the pressure from being isotropic such that the mean pressure is

$$p = \frac{2e^2\ddot{R}}{\kappa r^2R^3} - \frac{4m\ddot{R}}{\kappa rR^3} - \frac{2\ddot{R}}{\kappa R^3} - \frac{e^2\dot{R}^2}{\kappa r^2R^4} + \frac{2m\dot{R}^2}{\kappa rR^4} + \frac{\dot{R}^2}{\kappa R^4}. \tag{77}$$

$$G^a_a = \kappa(\mu - 3p) = \frac{12m\ddot{R}}{rR^3} - \frac{6e^2\ddot{R}}{r^2R^3} - \frac{12m\dot{R}}{r^2R^3} + \frac{6\ddot{R}}{R^3} \tag{78}$$

yields

$$\mu = -\frac{12m\dot{R}}{\kappa r^2R^3} - \frac{3e^2\dot{R}^2}{\kappa r^2R^4} + \frac{6m\dot{R}^2}{\kappa rR^4} + \frac{3\dot{R}^2}{\kappa R^4}. \tag{79}$$

In the radiation-dominated case

$$p = \frac{1}{\kappa R^2t^2} + \frac{2m}{\kappa rR^2t^2} - \frac{e^2}{\kappa r^2R^2t^2} \tag{80}$$

and

$$\mu = \frac{3}{\kappa R^2t^2} + \frac{6m}{\kappa rR^2t^2} - \frac{3e^2}{\kappa r^2R^2t^2} - \frac{12m}{\kappa r^2R^2t}. \tag{81}$$

The weak energy condition will be violated for

$$4mt > 2mr - e^2 + r^2, \tag{82}$$

and the dominant energy condition will be violated for

$$6mt > 2mr - e^2 + r^2, \tag{83}$$

which should not differ much from the uncharged case assuming $m/r \gg e^2/r^2$. In the matter-dominated case the average pressure is zero and

$$\mu = \frac{12}{\kappa R^2t^2} + \frac{24m}{\kappa rR^2t^2} - \frac{12e^2}{\kappa r^2R^2t^2} - \frac{24m}{\kappa r^2R^2t}, \tag{84}$$

so the weak energy condition is violated for

$$2mt > 2mr - e^2 + r^2, \tag{85}$$

which again should not significantly differ from the uncharged case. Due to the anisotropic pressure, the dominant energy condition may also be violated however.

6. New black hole solutions

Since Thakurta (1981) and Sultana and Dyer (2005) obtained different sources by doing a conformal transformation on different forms of the isolated black hole metric, then here we will perform a conformal transformation on the isotropic form of the black hole metric. Starting with the isotropic Schwarzschild metric

$$\mathrm{d}s^2 = -\left(\frac{1 - m/(2r)}{1 + m/(2r)}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{m}{2r}\right)^4 (\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)), \tag{86}$$

performing a conformal transformation to obtain a cosmological black hole

$$\mathrm{d}s^2 = [R(t)]^2 \left(-\left(\frac{1 - m/(2r)}{1 + m/(2r)}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{m}{2r}\right)^4 \left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right)\right), \tag{87}$$

and then making the transformation $\mathrm{d}t_c = R(t)\mathrm{d}t$, yields

$$\mathrm{d}s^2 = -\left(\frac{1 - m/(2r)}{1 + m/(2r)}\right)^2 \mathrm{d}{t_c}^2 + [R(t_c)]^2 \left(1 + \frac{m}{2r}\right)^4 \left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right), \tag{88}$$

which looks just like the McVittie metric except that the mass is constant. The event horizon occurs at $r = m/2$, since this is just a conformal transformation of the isotropic form of the Schwarzschild metric. The expansion of the universe scales the r dimension such that objects comoving with the expansion (remaining at fixed r) have their spatial separation increase with R, so the event horizon of the black hole grows with the expansion of the universe such that it appears to remain fixed in size in co-ordinates comoving with the universe's expansion. Thus, this metric appears to represent a cosmological black hole that expands with the universe, with the effective mass of the black hole appearing to remain the same in comoving co-ordinates or grow according to the expansion of the universe in non-comoving co-ordinates.

This metric has non-zero Einstein tensor components

$$G^0_0 = \frac{3(m+2r)^2\dot{R}^2}{(m-2r)^2R^2} \tag{89a}$$

$$G^0_1 = -\frac{8m(m+2r)\dot{R}}{(m-2r)^3R} \tag{89b}$$

$$G^1_0 = \frac{128mr^4\dot{R}}{(m+2r)^5(m-2r)R^3} \tag{89c}$$

$$G^1_1 = G^2_2 = G^3_3 = \left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right)\frac{(m+2r)^2}{(m-2r)^2}. \tag{89d}$$

Looking for a solution that consists of a perfect fluid plus a heat conduction component, then

$$G^a_a = \kappa(\mu - 3p) = \left(6\frac{\ddot{R}}{R} + 6\frac{\dot{R}^2}{R^2}\right)\frac{(m+2r)^2}{(m-2r)^2}. \tag{90}$$

Since $G^2_2 = G^3_3 = -\kappa p$,

$$p = -\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right)\frac{(m+2r)^2}{\kappa(m-2r)^2}. \tag{91}$$

Thus, the energy density is given by

$$\mu = \left(3\frac{\dot{R}^2}{R^2}\right)\frac{(m+2r)^2}{\kappa(m-2r)^2}. \tag{92}$$

For a radiation-dominated universe, $R = (2H_o t_c)^{1/2}$, which yields

$$p = \frac{(m+2r)^2}{4\kappa {t_c}^2(m-2r)^2} \tag{93}$$

and

$$\mu = \frac{3(m+2r)^2}{4\kappa {t_c}^2(m-2r)^2}. \tag{94}$$

Thus, the energy density is always positive and the pressure is one third of the energy density everywhere (just as it is for a radiation-dominated FLRW universe), so the energy conditions are satisfied everywhere. In the limit as r goes to zero or r goes to infinity, the energy density and pressure become that of the standard FLRW universe.

As the event horizon $r = m/2$ is approached from either side, the energy density and pressure both approach infinity. For a matter-dominated universe, $R = (3H_o t_c/2)^{2/3}$, which yields $p = 0$ and

$$\mu = \frac{4(m+2r)^2}{\kappa 3 {t_c}^2 (m-2r)^2}. \tag{95}$$

Thus, the energy density is positive everywhere, becoming infinite at the event horizon and approaching the energy density of a matter-dominated FLRW universe as r goes to zero or infinity, just as in the radiation-dominated case. The pressure is zero everywhere, just as it is for a matter-dominated FLRW universe. Since the energy density is always positive and the pressure is zero, the energy conditions are satisfied everywhere.

Extending this to the Reissner-Nordström case then

$$\mathrm{d}s^2 = -\left(\frac{1 - m^2/(4r^2) + e^2/(4r^2)}{(1+m/(2r))^2 - e^2/(4r^2)}\right)^2 \mathrm{d}{t_c}^2 + \\ [R(t_c)]^2 \left(\left(1+\frac{m}{2r}\right)^2 - \frac{e^2}{4r^2}\right)^2 (\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)). \tag{96}$$

This metric has non-zero Einstein tensor components

$$G^0_0 = \frac{3(e+m+2r)^2(e-m-2r)^2}{(e^2-m^2+4r^2)^2}\frac{\dot{R}^2}{R^2} + \frac{256e^2r^4}{(e+m+2r)^4(e-m-2r)^4R^2} \tag{97a}$$

$$G^0_1 = -\frac{8(e^2(m+4r) - m(m+2r)^2)(e+m+2r)(e-m-2r)}{(e^2-m^2+4r^2)^3}\frac{\dot{R}}{R} \tag{97b}$$

$$G^1_0 = \frac{128r^4(e^2(m+4r) - m(m+2r)^2)}{(e^2-m^2+4r^2)^2(e+m+2r)^3(e-m-2r)^3}\frac{\dot{R}}{R^3} \tag{97c}$$

$$G^1_1 = \frac{(e+m+2r)^2(e-m-2r)^2}{(e^2-m^2+4r^2)^2}\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right) + \frac{256e^2r^4}{(e+m+2r)^4(e-m-2r)^4R^2} \tag{97d}$$

$$G^2_2 = G^3_3 = \frac{(e+m+2r)^2(e-m-2r)^2}{(e^2-m^2+4r^2)^2}\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right) - \\ \frac{256e^2r^4}{(e+m+2r)^4(e-m-2r)^4R^2}. \tag{97e}$$

Looking for a solution that consists of a perfect fluid plus heat conduction component plus electric field, the terms that depend on the scale factor as $1/R^2$ correspond to the usual electric field components of an isolated isotropic Reissner-Nordström black hole

$$G^0_0 = G^1_1 = -G^2_2 = -G^3_3 = \frac{256e^2r^4}{(e+m+2r)^4(e-m-2r)^4}, \tag{98}$$

so these simply represent the electric field. Thus, the pressure is given by

$$p = -\frac{(e+m+2r)^2(e-m-2r)^2}{\kappa(e^2-m^2+4r^2)^2}\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right) \tag{99}$$

and the energy density is given by

$$\mu = \frac{3(e+m+2r)^2(e-m-2r)^2}{\kappa(e^2-m^2+4r^2)^2}\frac{\dot{R}^2}{R^2}. \tag{100}$$

In the case of a radiation-dominated universe, the pressure and energy density are given by

$$p = \frac{(e+m+2r)^2(e-m-2r)^2}{\kappa(e^2-m^2+4r^2)^2}\frac{1}{4t_c^2} \tag{101}$$

and

$$\mu = \frac{(e+m+2r)^2(e-m-2r)^2}{\kappa(e^2-m^2+4r^2)^2}\frac{3}{4t_c^2}, \tag{102}$$

and in the case of a matter-dominated universe, the pressure is zero and the energy density is given by

$$\mu = \frac{(e+m+2r)^2(e-m-2r)^2}{\kappa(e^2-m^2+4r^2)^2}\frac{4}{3t_c^2}. \tag{103}$$

The energy density is always positive. In the case of a radiation-dominated universe the pressure is always one third of the energy density, and in the case of a matter-dominated universe the pressure is always zero. Thus, the energy conditions are everywhere satisfied by this solution.

7. Discussion and summary

Despite M$^{\rm c}$Vittie's (1933) use of physical considerations to derive his spacetime and Thakurta's (1981) attempt to follow it by also deriving a spacetime with isotropic pressure, the M$^{\rm c}$Vittie and Thakurta spacetimes violate the energy conditions within the neighbourhood of the black hole, so the very part of the spacetime that is of interest is invalid. Also, it is not clear that the M$^{\rm c}$Vittie spacetime technically even represents a black hole, since the primary candidate for an event horizon is not one.

The Sultana and Dyer (2005) black hole spacetime becomes invalid for ever-increasing values of r with time, suggesting regions of spacetime that could be influenced by the black hole are incompatible with it. It would not be particularly useful to only be able to have a universe with a black hole in it only so long as the universe were not influenced by it. However, we have calculated that at a certain radius outside the event horizon, the critical surface between the valid and invalid regions of spacetime becomes timelike, so it is possible for inner regions to influence outer regions so that the solution is physically relevant in some regions of spacetime.

Of the previous spacetimes for asymptotically Einstein-de Sitter black holes, the Vaidya (1977) spacetime is the only one for which the invalid region of spacetime can always be made infinitesimally small, thanks to the arbitrary magnitude of the scale factor. Since it should theoretically be possible to set the scale factor to 1 at any time in the universe, then it seems artificial that the only way to prevent large regions of the spacetime from being unphysical is by choosing to set the scale factor to 1 in the first split second after the Big Bang. Also, the primary candidate for an event horizon is not one, so like the M$^{\rm c}$Vittie spacetime, the Vaidya spacetime may not even technically represent a black hole.

Unlike the previous spacetimes, the new cosmological black hole solutions presented in this paper neatly satisfy the energy conditions. The only oddity with the solutions is that the density (and the pressure in the radiation-dominated case) become infinite at the event horizon. A calculation of the Weyl scalar reveals that no singularity exists at the event horizon of the black hole, so the event horizon does not take on the role of the point mass singularity for a Schwarzschild black hole. The Weyl scalar is infinite only at the time of the Big Bang, so the infinite density at the event horizon must correspond to a finite total mass. The infinite density (and pressure) is perhaps not that unrealistic considering that the entire universe would have began that way, so the density (and pressure) would simply need to remain infinite at the event horizon from the time of the Big Bang.

It is interesting that the Thakurta, Sultana and Dyer, and new black hole spacetimes were all derived via a conformal transformation of an isolated black hole spacetime, the only difference being the original form of the metric for the black hole spacetime. The original forms of the black hole metric (standard Schwarzschild, Eddington-Finkelstein/Kerr-Schild, isotropic) are related by simple co-ordinate transformations, yet it is not possible for a simple transformation of co-ordinates to make the unphysical mass-energy distributions for the Thakurta and the Sultana and Dyer black holes look like the physical mass-energy distribution of the new black hole solutions. This suggests the difference must arise due to the action of the conformal transformation and the non-conformally invariant nature of the energy-momentum tensor. Thus, in obtaining new solutions via conformal transformations, it may be useful to make co-ordinate transformations of the original metric to obtain physically-different solutions.

Acknowledgments

We acknowledge a useful comment from Amanda Peet. This research was supported by an NSERC Post-Graduate Scholarship to MLM and an NSERC Discovery Grant to CCD.

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