Orbital Elements and Astrometric Orbit Determination

Catalina Sky Survey
Earth’s Ecliptic Plane

- Summer solstice
- Winter solstice
- Vernal equinox
- Autumnal equinox
- Path of Sun (Ecliptic)
- CE = 23.5°
- CE = 23.5°
- NEP
- SCP
- SEP
- NCP
Solar System Planets (in elliptical orbits)

Planetary orbits are inclined.

<table>
<thead>
<tr>
<th>Name</th>
<th>Inclination to ecliptic</th>
<th>Inclination to Sun's equator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>7.01°</td>
<td>3.38°</td>
</tr>
<tr>
<td>Venus</td>
<td>3.39°</td>
<td>3.86°</td>
</tr>
<tr>
<td>Earth</td>
<td>0°</td>
<td>7.155°</td>
</tr>
<tr>
<td>Mars</td>
<td>1.85°</td>
<td>5.65°</td>
</tr>
<tr>
<td>Jupiter</td>
<td>1.31°</td>
<td>6.09°</td>
</tr>
<tr>
<td>Saturn</td>
<td>2.49°</td>
<td>5.51°</td>
</tr>
<tr>
<td>Uranus</td>
<td>0.77°</td>
<td>6.48°</td>
</tr>
<tr>
<td>Neptune</td>
<td>1.77°</td>
<td>6.43°</td>
</tr>
</tbody>
</table>

Pluto = 17° 9'
Mercury = 7° 0'
Venus = 3° 24'
Saturn = 2° 29'
Mars = 1° 51'
Neptune = 1° 46'
Jupiter = 1° 18'
Uranus = 0° 46'

ANGL ES OF

ECLIPTIC = 0° = EARTH
Orbital Elements

So how many parameters (= orbital elements) do we need to describe a planetary orbit?

[Diagram of planetary orbital plane and reference plane (e.g., ecliptic, equatorial)]
Orbital Elements

So how many parameters (= orbital elements) do we need to describe a planetary orbit?

1. We need to know the size and shape of the elliptical orbit itself (2 elements).
2. We need to know the inclination and rotation of the orbit (2 elements).
3. We need a reference time and a reference point (2 elements).
Orbits

Pericenter
Perigee
Perihelion
Periastron

Central Body

Apocenter
Apogee
Aphelion
Apastron

\( \varepsilon \) and \( a \) are eccentricity and semi-major axis length, respectively.
Orbital Elements

1. Semi-major axis \( (a) \) and eccentricity \( (e) \).

2. Inclination \( (i) \), longitude of ascending node \( (\Omega; \text{location}) \), argument of perihelion \( (\omega; \text{rotation}) \): \( \Leftarrow \) 3 angles; \( \Omega \) and \( \omega \) are measured in the counter-clockwise direction and show the location of the node and rotation of the plane, respectively.

3. Epoch of perihelion \( (\tau) \).
Orbits

- $\Omega$ and $i$ measured in the ecliptic plane
- $\nu$ and $\omega$ measured in the orbital plane
- Angles Are Measured Counter Clockwise

North Ecliptic Pole

h = (0,0,h)

Descending node

Ascending node

Northern Vernal Equinox

Need a reference plane
## Orbital Elements

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<tr>
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<th>Symbol</th>
<th>Orbital element?</th>
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<tr>
<td>Semimajor axis [AU]</td>
<td>$a$</td>
<td>✓</td>
</tr>
<tr>
<td>Epoch of perihelion [Julian date]</td>
<td>$\tau$</td>
<td>✓</td>
</tr>
<tr>
<td>Current epoch [Julian date]</td>
<td>$t$</td>
<td></td>
</tr>
<tr>
<td>True anomaly [rad]</td>
<td>$\nu$</td>
<td></td>
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<tr>
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<td>$\omega$</td>
<td>✓</td>
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<td>Polar angle from the $x$-axis [rad]</td>
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<tr>
<td>Longitude of ascending node [rad]</td>
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<td>$M = n(t - \tau)$</td>
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<tr>
<td>Orbital period [days]</td>
<td>$p = 2\pi/n$</td>
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What if you want to know where the object is $\Delta t$ later?

Kepler’s 3$^{rd}$ law gives you the orbital period ($P$), but does not tell you where the object is at any given time.
Kepler's Equation and True, Eccentric, and Mean Anomaly

An object of an elliptical orbit with a non-uniform velocity is difficult to make a connection between the time and its position. Is there an easy solution?

Equation of the ellipse: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)

- \( a \) = semi-major axis
- \( b \) = semi-minor axis
- \( b^2 = a^2(1 - e^2) \)

\[
r = a\left(1 - e\cos E\right)
\]

- \( E \) = Angle Varying
- From 0 to 2\( \pi \)
- Pericenter: \( E = 0 \)
Kepler's Equation and True, Eccentric, and Mean Anomaly

**Anomaly**: An angular value used to describe the position of one member of a binary system w.r.t. the other.

**Eccentric Anomaly** \((E; \text{radian})\): If the position of the orbiting body is projected onto the reference circle as shown in the previous slide, the angular position of the reference position is called the eccentric anomaly, \(E\). The reference position when time = zero will be when the body is at periastron.

**True Anomaly** \((\nu; \text{radian})\): The angular position of the body from periastron as measured about the focus.

**Mean Anomaly** \((M)\): An angle that would have span if the ellipse is run with uniform velocity.

\[
M(t) = n t + M(0) = \left(\frac{2\pi}{T_{\text{rev}}}\right) t + M(0)
\]

\[
M = n(t - \tau) = E - e \sin(E)
\]

(See the lab document)

\(\leftarrow E\) in radian, \(\tau = \) periastron passage time; gives relation between the time and position; Kepler’s eqn.)
Kepler's Equation and True, Eccentric, and Mean Anomaly

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\[ r = a \left( 1 - e \cos E \right) \]

Mean Anomaly

\[ M = n(t - \tau) = E - e \sin E \]

True Anomaly

\[ v = 2 \arctan \left( \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2} \right) \]

\( E \) = Eccentric Anomaly
\( n \) = Orbital Period
\( P = 2\pi/n \)
\( \tau \) = Epoch of Pericenter
What if you want to know where the object is $\Delta t$ later?

$$M = n(t - \tau) = E - e \sin E$$

Determine $M$ and solve transcendental equation

$$v = 2 \arctan \left[ \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \right]$$
Orbital Elements

- Vernal point
- Ascending node
- Gravitational Constant $k$

**Gravitational Constant $k$**

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How do we determine orbits of solar system objects?
So how do we determine the 6 orbital elements?

What kind of information do we need to determine the 6 orbital elements?
Orbital elements determination = Having information of the position and velocity of an object at any given time.
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Mathematically, you need 6 independent measurements. If you know the acceleration, you can calculate the velocity.
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Mathematically, you need 6 independent measurements. If you know the acceleration, you can calculate the velocity.

(Projected) Positions of an object at three different epoch would do the trick? (So this gives 6 measurements.)
First, let’s remind this:

Consider the orbital motion of masses $m_1$ and $m_2$, located at $\mathbf{r}_1$ and $\mathbf{r}_2$ with respect to the center of mass of the system. If no external forces act, then by conservation of momentum the center of mass is fixed, i.e.,

$$\mathbf{r}_{CM} = \frac{(m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2)}{(m_1 + m_2)} = 0,$$

(1)

and $\mathbf{r}_1 = -\frac{m_2 \mathbf{r}}{(m_1 + m_2)}$ and $\mathbf{r}_2 = \frac{m_1 \mathbf{r}}{(m_1 + m_2)}$, where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is the vector jointing the two objects.

Each mass experiences the gravitational force of the other, and the relative motion of the two is described by

$$\ddot{\mathbf{r}} = -k^2 \frac{\mathbf{r}}{r^3},$$

(2)

where dots denote time derivatives. The constant $k^2 = G(m_1 + m_2)$ and $G$ is Newton’s constant. In this application we consider the case where $m_1$ is the mass of the sun and $m_1 \gg m_2$.

Figure 1: Two masses (red circles) orbit about the center of mass (black dot) at positions defined by $\mathbf{r}_1$ and $\mathbf{r}_2$. 
Basic geometry and equations

\[ r = R + \rho s \]

\[ r^2 = \rho^2 + R^2 + 2\rho R \cdot s \]

\[ r = (x, y, z) \]

\[ R = (X, Y, Z) \]

\( \rho \): distance from Earth to asteroid; 
\( s \): positional unit vector

- Meaning of “measurement of the position of the asteroid”?
- \( R \) from where?

Figure 6: The relative position of the sun, earth (observer), and the target asteroid.

\[ \ddot{r} = -k^2 \frac{r}{r^3} \quad \text{and} \quad \ddot{R} = -k^2 \frac{R}{R^3}. \]

\[ \ddot{r} = \ddot{R} + \ddot{\rho} s + 2\ddot{\rho} \dot{s} + \rho \ddot{s}. \]

\[ s \left( \dddot{\rho} + k^2 \frac{\rho}{r^3} \right) + 2 \dddot{\rho} \dot{s} + \rho \dddot{s} = k^2 R \left( \frac{1}{R^3} - \frac{1}{r^3} \right). \]

\[ \dot{\rho} = \frac{k^2}{2} \left( \frac{1}{R^3} - \frac{1}{r^3} \right) \frac{\dot{s} \cdot (R \times s)}{\dot{s} \cdot (\dot{s} \times s)}. \]

(See the lab document)
Basic geometry and equations

What we know:
- Three sky positions in sky;
- We do not know distance to the object!

Figure 6: The relative position of the sun, earth (observer), and the target asteroid.

- Start with $R$ and $s$
- One can estimate $\rho$ and $r$ from $R$ and $s$. (↔ Solve iteratively for $\rho$, given an initial guess for $r$)

Example:
- Initial guess for $r$ (e.g., 2AU);
- Calculate $\rho$
- Calculate $r$
- Are they converging?
Three positional measurements

Measurements at three different epochs of $t_1$, $t_2$, & $t_3$.

Determination of the position requires one measurement, velocity requires two measurements, and acceleration does three measurements.

Thus, if $\tau_1 = t_2 - t_1$ and $\tau_3 = t_3 - t_2$ then

$$s_1 = s_2 - \tau_1 \dot{s}_2 + \frac{1}{2} \tau_1^2 \ddot{s}_2,$$

$$s_3 = s_2 + \tau_3 \dot{s}_2 + \frac{1}{2} \tau_3^2 \ddot{s}_2,$$

yielding

$$\dot{s}_2 = \frac{\tau_3 (s_2 - s_1)}{\tau_1 (\tau_1 + \tau_3)} + \frac{\tau_1 (s_3 - s_2)}{\tau_3 (\tau_1 + \tau_3)},$$

$$\ddot{s}_2 = \frac{2(s_3 - s_2)}{\tau_3 (\tau_1 + \tau_3)} - \frac{2(s_2 - s_1)}{\tau_1 (\tau_1 + \tau_3)}.$$
Three sky position measurements: Laplace’s method

1. Calculate \( \frac{ds}{dt} \), \( \frac{d^2s}{dt^2} \) and \( \rho \);

2. Calculate \( r \) and \( v \) at \( t_2 \);

3. Calculate all other orbital elements.
   (See the lab document for this. Examples of Ceres and 10 Hygeia are given.)

Once you have computed the orbital elements, are they the same forever (= converging)?
Near Earth Objects (NEO)
About ~1650 known Potentially Hazardous NEOs
Near-misses by Known Asteroids in the 21st Century

Solid lines indicate nominal asteroid trajectory.
Dashed lines indicate current trajectory uncertainty.

Earth

Orbit of the Moon

Geosynchronous satellites

Why so many more fly-bys before 2013?
Because with our current facilities,
we often discover near-miss asteroids
just before or just after fly-by.

Color and size of circles indicate approximate relative size of each asteroid.
The largest asteroid shown,
1997 XF11, is roughly 1 mile across.

Visualization by Alex H. Parker
Data collected on Feb 3, 2013 from
http://neo.jpl.nasa.gov/cgi-bin/neo_ca
400-m behemoth that passed only 0.85 lunar radii from Earth in 2011!

Very low albedo (~0.04)
NEO Challenges

• Don’t know where to look
  – Need to carry out large, regular surveys of the sky

• Computationally intensive to find them
  – These surveys generate intense amounts of data, and you need to search for a needle in a haystack

• Very faint in the visible
  – Because of their low albedos, they are very faint and only are observable until they are closer

• Hard to estimate their size
  – Because you can only measure their brightness and distance, you would need to know the composition (i.e. albedo) to estimate their size

Figure-of-merit (Étendue):
Collecting Area of Telescope x Field-of-View
Pan-STARRS Project

- 2x 1.8-meter Telescopes with 3 degree fields-of-view
  - 1.4 gigapixel camera
  - 2 gigabytes per image!

- Observe 3000 square degrees per night

- One image taken every ~2 minutes
  - 5 terabytes of data per night

- Final version will have four identical telescopes
Large Synoptic Survey Telescope

- 8.4-meter telescopes with 3.5 degree field-of-view
  - 3 gigapixel camera
- Takes images every 20 seconds!
  - 30 TB/night
- Online in 2019
Sentinel Space Telescope (Proposed)

Asteroids easier to see in the infrared

Wide-field Infrared (7-12 µm) Telescope

NEOWISE, a repurposed space satellite, is already doing some of this work