AST 222 Winter 2011

Assignment #5

Due 12pm, Apr 4

This assignment is due before class on Monday, Apr 4; that is, at 12:10 **sharp**. Assignments handed in late — including at the end of class — will lose 20% credit. Answers should be posted online soon after; once answers are posted, of course, no further assignments will be accepted.

Assignments may be handed in in person at class or by email. The instructor will have online office hours at 6pm on Sunday, Apr 3, and TA office hours will be on Tuesdays and Thursdays. Quesions may be asked via email at `ast222@astro.utoronto.ca`.

Show your work, and good luck!

Question 1 - A Multi-Component Universe [35 pts]

In this assignment, you will derive the multi-component versions of the equations you derived last assignment for a matter-only Universe,

$$\begin{aligned} \Omega &= 1 + \frac{\Omega_0 - 1}{1 + \Omega_0 z} \\ H &= H_0(1+z)(1+\Omega_0 z)^{1/2}. \end{aligned}$$

1 (a)

(10 pts) Starting with

$$H^2(1-\Omega)R^2 = H_0^2(1-\Omega_0),$$

derive the evolution of the Hubble constant with z,

$$H = H_0(1+z)\left[\Omega_{m,0}(1+z) + \Omega_{r,0}(1+z)^2 + \Omega_{\Lambda,0}(1+z)^{-2} + (1-\Omega_0)\right]^{1/2}.$$

You'll have to write Ω as $\Omega_m + \Omega_r + \Omega_\Lambda$, and use the definition of Ω and the evolution of the density of the various components to express these in terms of their current-day values.

$$\begin{aligned}
H^2(1-\Omega)R^2 &= H_0^2(1-\Omega_0) \\
H^2(1-\Omega_m-\Omega_r-\Omega_\Lambda)R^2 &= H_0^2(1-\Omega_0) \\
H^2\left[1 - \frac{1}{\rho_{cr}}\left(\rho_m+\rho_r+\rho_\Lambda\right)\right]R^2 &= H_0^2(1-\Omega_0) \\
H^2\left[1 - \frac{8\pi G}{3H^2}\left(\rho_m+\rho_r+\rho_\Lambda\right)\right]R^2 &= H_0^2(1-\Omega_0) \\
H^2\left[1 - \frac{H_0^2}{H^2}\frac{8\pi G}{3H_0^2}\left(\rho_{m,0}(1+z)^3+\rho_{r,0}(1+z)^4+\rho_{\Lambda,0}\right)\right]R^2 &= H_0^2(1-\Omega_0) \\
H^2\left[1 - \frac{H_0^2}{H^2}\frac{1}{\rho_{cr,0}}\left(\rho_{m,0}(1+z)^3+\rho_{r,0}(1+z)^4+\rho_{\Lambda,0}\right)\right]R^2 &= H_0^2(1-\Omega_0) \\
H^2\left[1 - \frac{H_0^2}{H^2}\left(\Omega_{m,0}(1+z)^3+\Omega_{r,0}(1+z)^4+\Omega_{\Lambda,0}\right)\right]R^2 &= H_0^2(1-\Omega_0)
\end{aligned}$$

$$\begin{aligned} H^2(1+z)^{-2} - H_0^2\left(\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0}\right)(1+z)^{-2} &= H_0^2(1-\Omega_0) \\ H^2(1+z)^{-2} - H_0^2\left(\Omega_{m,0}(1+z) + \Omega_{r,0}(1+z)^2 + \Omega_{\Lambda,0}(1+z)^{-2}\right) &= H_0^2(1-\Omega_0) \end{aligned}$$

Finally, moving everything over to the right hand side,

$$\begin{aligned} H^2 &= H_0^2(1+z)^2\left[(1-\Omega_0) + \Omega_{m,0}(1+z) + \Omega_{r,0}(1+z)^2 + \Omega_{\Lambda,0}(1+z)^{-2}\right] \\ H^2 &= H_0^2(1+z)^2\left[(1-\Omega_0) + \Omega_{m,0}(1+z) + \Omega_{r,0}(1+z)^2 + \Omega_{\Lambda,0}(1+z)^{-2}\right] \\ H &= H_0(1+z)\left[(1-\Omega_0) + \Omega_{m,0}(1+z) + \Omega_{r,0}(1+z)^2 + \Omega_{\Lambda,0}(1+z)^{-2}\right]^{1/2} \end{aligned}$$

1 (b)

(10 pts) Now, derive the equivalent of

$$\Omega = 1 + \frac{\Omega_0 - 1}{1 + \Omega_0 z}$$

for a multicomponent Universe,

$$\Omega = 1 + \frac{(\Omega_0 - 1)}{(1-\Omega_0) + \Omega_{m,0}(1+z) + \Omega_{r,0}(1+z)^2 + \Omega_{\Lambda,0}(1+z)^{-2}}$$

. Note that although the denominator is much more complicated, it always remains nonzero, so that we maintain the property that if the Universe ever has $\Omega = 1$, it remains 1.

Start with the definition of Ω, and substitute the expression you've derived for $H(z)$ above.

$$\begin{aligned} \Omega &= \frac{\rho}{\rho_c} \\ &= \frac{8\pi G\rho}{3H^2} \\ &= \frac{8\pi G}{3}\frac{\rho_m + \rho_r + \rho_\Lambda}{H_0^2(1+z)^2\left[(1-\Omega_0) + \Omega_{m,0}(1+z) + \Omega_{r,0}(1+z)^2 + \Omega_{\Lambda,0}(1+z)^{-2}\right]} \\ &= \frac{8\pi G}{3H_0^2}\frac{\rho_{m,0}(1+z)^3 + \rho_{r,0}(1+z)^4 + \rho_{\Lambda,0}}{(1+z)^2\left[(1-\Omega_0) + \Omega_{m,0}(1+z) + \Omega_{r,0}(1+z)^2 + \Omega_{\Lambda,0}(1+z)^{-2}\right]} \\ &= \frac{1}{\rho_{cr,0}}\frac{\rho_{m,0}(1+z)^3 + \rho_{m,0}(1+z)^4 + \rho_{\Lambda,0}}{\left[(1-\Omega_0)(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0}\right]} \\ &= \frac{\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0}}{(1-\Omega_0)(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0}} \\ &= \frac{(1-\Omega_0)(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0} - (1-\Omega_0)(1+z)^2}{(1-\Omega_0)(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0}} \\ &= 1 + \frac{(\Omega_0 - 1)(1+z)^2}{(1-\Omega_0)(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0}} \\ &= 1 + \frac{(\Omega_0 - 1)}{(1-\Omega_0) + \Omega_{m,0}(1+z) + \Omega_{r,0}(1+z)^2 + \Omega_{\Lambda,0}(1+z)^{-2}} \end{aligned}$$

This is very similar to (but more complicated than!) the matter-only expression you derived in the last assignment,

$$\Omega = 1 + \frac{\Omega_0 - 1}{1 + \Omega_0 z}$$

and has the same consequence with respect to the special point $\Omega_0 = 1$; if Ω is ever one than it is always one.

1 (c)

(5 pts) The acceleration equation, too, changes its behaviour in a multicomponent universe. Starting with the acceleration equation,

$$\frac{d^2R}{dt^2} = \left\{ -\frac{4}{3}\pi G \left[\rho + \frac{3P}{c^2} \right] \right\} R,$$

find the scale factor, and redshift, at which the Universe's expansion switches from decelerating to accelerating. This happens recently in the Universes' evolution, where the contribution of radiation may be neglected. Use $\Omega_{\Lambda,0} = 0.742$ and $\Omega_{m,0} = 0.256$.

Here we're looking at the acceleration equation, and in particular finding out where the acceleration becomes zero. We can use it in the format provided, and (as suggested) neglect radiation:

$$\begin{aligned}
\frac{d^2R}{dt^2} &= \left\{ -\frac{4}{3}\pi G \left[\rho_m + \rho_r + \frac{3P_r}{c^2} \right] + \frac{1}{3}\Lambda c^2 \right\} R \\
0 &= \left\{ -\frac{4}{3}\pi G \left[\rho_m \right] + \frac{1}{3}\Lambda c^2 \right\} R \\
\Lambda c^2 &= 4\pi G \rho_m \\
\Lambda c^2 &= 4\pi G \rho_{m,0} (1+z)^3 \\
(1+z)^3 &= \frac{\Lambda c^2}{4\pi G \rho_{m,0}}
\end{aligned}$$

We have enough to plug in numbers here but we can simplify things further:

$$\begin{aligned}
(1+z)^3 &= \frac{8\pi G \rho_\Lambda}{4\pi G \rho_{m,0}} = \frac{8\pi G \rho_{\Lambda,0}}{4\pi G \rho_{m,0}} \\
&= \frac{2\rho_{\Lambda,0}}{\rho_{m,0}} \\
&= \frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}} \\
(1+z) &= \left(\frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}} \right)^{1/3} \\
R &= \left(\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}} \right)^{1/3} \\
R &= 0.570
\end{aligned}$$

So the Universe has been accelerating since $R = 0.570$ or $z = 0.755$.

1 (d)

(5 pts) A cosmological constant, or vacuum energy, has the property that it's energy density is constant. Does that mean that Ω_Λ is constant in a multicomponent Universe?

No; $\Omega_\Lambda = \frac{8\pi G \rho_\Lambda}{3H^2}$, and so while ρ_Λ is constant, H varies, and the quotient as a whole may vary.

(However, it can be shown that in a Λ-only or -dominated Universe, H is constant; so in a Universe where nothing else mattered, Ω_Λ would in fact be constant. But that's a particular case.)

1 (e)

(5 pts) Find the boundaries in redshift and scale factor between "radiation-dominated", "matter dominated", and "Lambda-dominated" phases of the Universes' evolution; that is, find at what redshifts and

scale factors matter first contributed as much to the expansion equation as radiation, and where a cosmological constant or vacuum energy contributed as much as mass. Use $\Omega_{\Lambda,0} = 0.742$, $\Omega_{m,0} = 0.256$, and $\Omega_{r,0} = 8.2 \times 10^{-5}$.

The contribution to the expansion equation is through ρ; thus we must find out when $\rho_r = \rho_m$, or equivalently when $\rho_r/\rho_m = 1$, and likewise when $\rho_m/\rho_\Lambda = 1$.

For matter-radiation equality, eg the boundary between radiation-domination and matter-domination,

$$\begin{aligned}
1 &= \frac{\rho_r}{\rho_m} \\
&= \frac{\rho_{r,0}R^{-4}}{\rho_{m,0}R^{-3}} \\
&= \frac{\rho_{r,0}}{\rho_{m,0}}R^{-1} \\
&= \frac{\rho_{r,0}/\rho_{cr,0}}{\rho_{m,0}/\rho cr, 0}R^{-1} \\
&= \frac{\Omega_{r,0}}{\Omega_{m,0}}R^{-1} \\
R &= \frac{\Omega_{r,0}}{\Omega_{m,0}} \\
&= \frac{8.2 \times 10^{-5}}{0.256} \\
&= 3.2 \times 10^{-4} \\
z &= \frac{1}{R} - 1 \\
&= 3120.
\end{aligned}$$

Likewise for matter-Λ equality:

$$\begin{aligned}
1 &= \frac{\rho_m}{\rho_\Lambda} \\
&= \frac{\rho_{m,0}R^{-3}}{\rho_{\Lambda,0}} \\
&= \frac{\rho_{m,0}}{\rho_{\Lambda,0}}R^{-3} \\
&= \frac{\rho_{m,0}/\rho_{cr,0}}{\rho_{\Lambda,0}/\rho cr, 0}R^{-2} \\
&= \frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}R^{-3} \\
R &= \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \\
&= \left(\frac{0.256}{.742}\right)^{1/3} \\
&= 0.701 \\
z &= \frac{1}{R} - 1 \\
&= 0.426.
\end{aligned}$$

The time of matter-Λ equality was very recently, cosmologically speaking. We are not really yet in a Λ-dominated era; $\rho_\Lambda \approx 3\rho_m$, so matter remains important.

Question 2 - Big-Bang Nucleosynthesis [25 pts]

2 (a)

(5 pts) In class, we said that the reactions converting neutrons to protons eventually "froze-out" at about $T = 10^{10}\,K$. One of the reasons for this was a relative paucity of electrons and positrons necessary to drive the weak interactions. At higher temperatures, typical photons were so energetic that they could create electron-positron pairs (that is, their energy was larger than twice the electron rest energy $E_\gamma > 2m_ec^2$. These pairs would last for a while, possibly reacting, but eventually they or another pair would find each other and annhilate. Below this energy, these pairs were no longer being replenished, and the neutrons stoped being converted to protons.

Demonstrate that this process actually does happen at about the temperature specified. The characteristic energy of a photon of a given blackbody with temperature T is $E_\gamma = kT$; from this, estimate the temperature at which electron-positron pairs can no longer be created.

Pairs can be created when:

$$\begin{aligned} E_\gamma &> 2m_ec^2 \\ kT &> 2m_ec^2 \\ T &> 2\frac{m_ec^2}{k} \\ &> 1.18 \times 10^{10}\,K \end{aligned}$$

At temperatures above $1.18 \times 10^{10}\,K$, these pairs can be routinely created; below, they cannot.

2 (b)

(10 pts) Once deuterium can be formed (at about $10^9\,K$, we have said that the neutrons quickly all go into deuterium, and thence on Helium. Here you'll demonstrate that there is plenty of time for these reactions to take place.

The number of collisions we would expect a neutron to make with protons over a period of time Δt is $n_p\sigma v\Delta t$ where n_p is the number density of protons, σ is the cross-section of interactions with a proton, and v is the velocity of the neutrons. Estimate the number density of protons at temperature $10^9\,K$ (about 87.5% of baryonic mass is in the form of protons; the current baryonic density is $\rho_b = 4.5 \times 10^{-28}\,\mathrm{kg\,m^3}$, and the temperature of the Universe at the current time is $T = 2.725\,K$. Estimate the velocity dispersion of neutrons at this temperature from kinetic theory, and take $\sigma = 4\pi r_n^2$ where $r_n \approx 10^{-15}\,\mathrm{m}$. The amount of time available to the neutrons was the characteristic time for expansion at that time, $1/H = (2t) \approx 356$ sec.

Calculate the expected number of collisions over this time. If this number is significantly greater than 1, then the neutron would likely have collided and reacted with a proton in this time.

$$\begin{aligned} N_{\mathrm{col}} &= n_p\sigma v\Delta t \\ n_{p,0} &= \frac{0.875\rho_{b,0}}{m_p} \\ &= 0.235\,\mathrm{m}^{-3} \\ n_p &= n_{p,0}R^{-3} \\ &= n_{p,0}\left(\frac{T_0}{T}\right)^{-3} \\ &= 0.235\,\mathrm{m}^{-3} \times 4.94 \times 10^{25} \\ &= 1.16 \times 10^{25}\,\mathrm{m}^{-3} \\ \sigma &= 4\pi(10^{-15}\,\mathrm{m})^2 \\ &= 1.26 \times 10^{-29}\,\mathrm{m}^2 \end{aligned}$$

$$\begin{aligned}
\frac{1}{2}m_n v^2 &= \frac{3}{2}kT \\
v &= \sqrt{\frac{3kT}{m_n}} \\
&= 4.98 \times 10^6 \text{ m sec}^{-1} \\
N_{\text{col}} &= \left(1.16 \times 10^{25}\text{ m}^{-3}\right)\left(1.26 \times 10^{-29}\text{ m}^2\right)\left(4.98 \times 10^6\text{ m sec}^{-1}\right)(356\text{ sec}) \\
&= 2.59 \times 10^5 \\
&\gg 1
\end{aligned}$$

There was more than enough time for all the neutrons to collide with protons and form deuterium.

2 (c)

(10 pts) The physics of big-bang nucleosynthesis – and, for that matter, the cosmic microwave background – were worked out long before the standard model included a cosmological constant or vacuum energy term. Show that the introduction of this term does not make a significant difference in either the expansion or the acceleration equations at the time of BBN by showing that $\rho_r \gg \rho_\Lambda$ (or, equivalently, $\rho_r/\rho_\Lambda \gg 1$ and similarly $P_r \gg P_\Lambda$. Use $\Omega_{\Lambda,0} = 0.742$ and $\Omega_{r,0} = 8.2 \times 10^{-5}$.

We know how both components behave as a function of scale factor (or, equivalently, of temperature, since $T \propto R^{-1}$, so:

$$\begin{aligned}
\frac{\rho_r}{\rho_\Lambda} &= \frac{\rho_{r,0}R^{-4}}{\rho_{\Lambda,0}} \\
&= \frac{\rho_{r,0}}{\rho_{\Lambda,0}}\left(\frac{T}{T_0}\right)^4 \\
&= \frac{\rho_{r,0}/\rho_{cr,0}T^4}{\rho_{\Lambda,0}/\rho_{cr,o}T_0^4} \\
&= \frac{\Omega_{r,0}}{\Omega_{\Lambda,0}}\left(\frac{T}{T_0}\right)^4 \\
&= 1.11 \times 10^-4\left(\frac{T}{T_0}\right)^4 \\
&= 1.11 \times 10^-4 \times 1.81 \times 10^{34} \\
&= 2.01 \times 10^{30}
\end{aligned}$$

Radiation was well and truly dominant at this time.

Pressure can be easily calculated from this:

$$\begin{aligned}
P_r &= \frac{1}{3}\rho_r c^2 \\
P_\Lambda &= -\rho_\Lambda c^2 \\
\frac{P_r}{P_\Lambda} &= \frac{\frac{1}{3}\rho_r c^2}{-\rho_\Lambda c^2} \\
&= -\frac{1}{3}\frac{\rho_r}{\rho_\Lambda} \\
&= -6.71 \times 10^{29}.
\end{aligned}$$

So one can flatly say that Λ did not contribute at the time of BBN. A similar calculation can be done at the time of the CMB ($T = 3000\,K$); the numbers are less dramatic, but one can easily see that the numbers are still 2×10^6 and -6.71×10^5; well under a tenth of a percent contribution.

Question 3 - Horizon Distance, Connected Regions, and Structure (40 pts)

The horizon distance from a point is the distance to the furthest regions causally connected to that point — basically where a photon could have reached us over this history of the Universe. This can be calculated by

$$d_h(t) = R(t)\int_0^t \frac{cdt'}{R(t')}.$$

That is to say, since the beginning of time, a photon has travelled at the speed of light, and over each dt' the Universe had a scale factor of $R(t')$; that has to be scaled up to a factor of $R(t)$, or the scale factor at the time we're calculating the horizion distance.

3 (a)

(5 pts) In the previous assignment, you showed that in a flat, matter dominated Universe, $R(t) \propto t^{2/3}$. Perform the same calculation for a radiation dominated Universe, and show $R(t) \propto t^{1/2}$.

We can follow our derivation from Assignment 4 almost exactly with the only difference being how $\rho(R)$ behaves:

$$\begin{aligned}
\left[H^2 - \frac{8}{3}\pi G\rho\right]R^2 &= -kc^2 \\
\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right]R^2 &= -kc^2 \\
\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right]R^2 &= H_0\left(1-\Omega_0\right) \\
\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right]R^2 &= 0 \\
\left(\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho R^2 &= 0 \\
\left(\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho_{r,0}R^{-4}R^2 &= 0 \\
\left(\frac{dR}{dt}\right)^2 &= \frac{8}{3}\pi G\rho_{r,0}R^{-2} = H_0^2\Omega_{r,0}R^{-1} = H_0^2\Omega_{r,0}R^{-1} \\
\frac{dR}{dt} &= H_0\Omega_{r,0}^{1/2}R^{-1} \\
RdR &= H_0\Omega_{r,0}^{1/2}dt \\
\int_0^R R'dR' &= H_0\Omega_{r,0}^{1/2}\int_0^t dt' \\
\frac{1}{2}R(t)^2 &= H_0\Omega_{r,0}^{1/2}t \\
R(t) &= \left(2H_0\Omega_{r,0}^{1/2}t\right)^{1/2} \\
t(R) &= \frac{1}{2H_0\Omega_{r,0}^{1/2}}R^2.
\end{aligned}$$

Note re-doing the expresion from assignment 4 for $R(t)$ in a matter-dominated phase in a multi-component Universe gives us a similar result,

$$R(t) = \left(\frac{3}{2}H_0\Omega_{m,0}^{1/2}t\right)^{2/3}.$$

Because the Universe is no multi-component, a flat Universe no longer implies $\Omega_{m,0}$ or $\Omega_{r,0}$ equal 1, even though they Ω_m and Ω_r would have been approximately one in the matter- and radiation-dominated eras of a flat universe respectively.

3 (b)

(5 pts) Using the definition above and the expressions from part (a), show that (since our Universe can be considered flat), during the radiation era the horizon distance was

$$d_h(t) = 2ct$$

and in the matter era, the horizon distance was

$$d_h(t) = 3ct.$$

From our definition of the horizon distance, for a flat radiation-dominated Universe,

$$\begin{aligned} d_h(t) &= R(t)\int_0^t \frac{cdt'}{R(t')} \\ &= \left(2H_0\Omega_{r,0}^{1/2}t\right)^{1/2}\int_0^t \frac{cdt'}{\left(2H_0\Omega_{r,0}^{1/2}t'\right)^{1/2}} \\ &= t^{1/2}\int_0^t ct'^{-1/2}dt' \\ &= t^{1/2}\left[2ct'^{1/2}\right]_0^t \\ &= 2ct, \end{aligned}$$

and for a matter-dominated Universe

$$\begin{aligned} d_h(t) &= R(t)\int_0^t \frac{cdt'}{R(t')} \\ &= \left(\frac{3}{2}H_0\Omega_{m,0}^{1/2}t\right)^{2/3}\int_0^t \frac{cdt'}{\left(\frac{3}{2}H_0\Omega_{m,0}^{1/2}t'\right)^{2/3}} \\ &= t^{2/3}\int_0^t ct'^{-2/3}dt' \\ &= t^{2/3}\left[3ct'^{1/3}\right]_0^t \\ &= 3ct. \end{aligned}$$

3 (c)

(10 pts) In the radiation era, how does the total mass/energy of the Universe within a connected region vary with time? And in the matter era?

Note that there's a couple ways you could define "connected region", and I didn't specify exactly what I meant by this. A region where every point is connected with every other point is a sphere of *diameter* d_h; the region connected to a single point is a sphere of *radius* d_h. I'll use the larger volume here in these answers; the other answer would be fine, too, and the masses are simply smaller by a factor of 8.

The mass of the connceted region in the radiation-domianted era can be found by

$$\begin{aligned} M(t) &= \frac{4}{3}\pi d_h(t)^3\rho(t) \\ &= \frac{4}{3}\pi\left(2ct\right)^3\rho_{r,0}R^{-4} \\ &= \frac{32}{3}\pi c^3t^3\rho_{r,0}\left(2H_0\Omega_{r,0}^{1/2}t\right)^{-2} \\ &= \frac{8}{3}\frac{\pi c^3}{H_0^2\Omega_{r,0}}\rho_{r,0}t \\ &= \frac{c^3}{G}t. \end{aligned}$$

Similarly, in the matter-dominated era,

$$\begin{aligned} M(t) &= \frac{4}{3}\pi d_h(t)^3\rho(t) \\ &= \frac{4}{3}\pi\left(3ct\right)^3\rho_{m,0}R^{-3} \\ &= 36\pi c^3t^3\rho_{m,0}\left(\frac{3}{2}H_0t\Omega_{m,0}^{1/2}\right)^{-2} \\ &= 16\frac{\pi c^3}{H_0^2\Omega_{m,0}}\rho_{m,0}t \\ &= \frac{6c^3}{G}t. \end{aligned}$$

In both cases, the total amount of mass causally connected increases linearly with time.

3 (d)

(10 pts) Find the size of the largest causally connected region of the CMB (z = 1100). What was the baryonic mass contained within that region? How about the time of BBN, $T = 10^9\ K$? Assume both of these events happen in the radiation era (the CMB just barely so). Carrol and Ostlie gives as relevant times $t_{\rm cmb} = 3.79 \times 10^5$ yr and $t_{\rm bbn} = 178$ sec.

We have, from the radiation-dominated result above, for the CMB:

$$\begin{aligned} M(t) &= \frac{8}{3}\frac{\pi c^3}{H_0^2}\rho_{r,0}t \\ &= \frac{8}{3}\frac{\pi G\rho_r,0}{H_0^2}\frac{c^3}{G}t \\ &= \frac{c^3}{G}t \\ &= \frac{c^3}{G}3.79\times 10^5\ \mathrm{yr} \\ &= 2.42\times 10^{18}\ M_\odot. \end{aligned}$$

as the total mass/energy. Similarly, the total mass/energy enclosed in a connected region for the BBN,

$$\begin{aligned} M(t) &= \frac{c^3}{G}\times 178\ \mathrm{sec} \\ &= 3.6\times 10^6\ M_\odot. \end{aligned}$$

Now the question is what fraction of this is baryonic matter. This fraction is

$$\begin{aligned}\frac{\rho_b}{\rho} &= \frac{\rho_b}{\rho_r + \rho_m + \rho_\Lambda} \\ &= \frac{\rho_{b,0} R^{-3}}{\rho_{r,0} R^{-4} + \rho_{m,0} R^{-3} + \rho_{\Lambda,0}}\end{aligned}$$

in the radiation-dominated regime, this is simply

$$\begin{aligned}\frac{\rho_b}{\rho} &= \frac{\rho_{b,0} R^{-3}}{\rho_{r,0} R^{-4}} \\ &= \frac{\Omega_{b,0} R^{-3}}{\Omega_{r,0} R^{-4}} \\ &= \frac{\Omega_{b,0}}{\Omega_{r,0}} R.\end{aligned}$$

In the matter-dominated regime,

$$\frac{\rho_b}{\rho} = \frac{\Omega_{b,0}}{\Omega_{m,0}},$$

or if one wanted more accurate results for the CMB era, one could include both matter and radiation,

$$\frac{\rho_b}{\rho} = \frac{\Omega_{b,0} R}{\Omega_{r,0} + \Omega_{m,0} R}.$$

Using the radiation-dominated era results to be consistent, we have

$$\begin{aligned}\left.\frac{\rho_b}{\rho}\right|_{\text{CMB}} &= \frac{\Omega_{b,0}}{\Omega_{r,0}} R_{\text{CMB}} \\ &= \frac{\Omega_{b,0}}{\Omega_{r,0}} \left(\frac{T_0}{T_{\text{CMB}}}\right) \\ &= \frac{0.046}{8.24 \times 10^5} \frac{2.725}{3000} \\ &= 0.507 \\ M_b(t_{\text{CMB}}) &= 2.42 \times 10^{18}\, M_\odot \times 0.507 \\ &= 1.22 \times 10^{18}\, M_\odot \\ \left.\frac{\rho_b}{\rho}\right|_{\text{BBN}} &= \frac{\Omega_{b,0}}{\Omega_{r,0}} R_{\text{BBN}} \\ &= \frac{0.046}{8.24 \times 10^5} \frac{2.725}{10^9} \\ &= 1.52 \times 10^{-6} \\ M_b(t_{\text{BBN}}) &= 3.6 \times 10^6\, M_\odot \times 1.52 \times 10^{-6} \\ &= 5.47\, M_\odot\end{aligned}$$

At these times, relatively small pieces of the Universe were actually in causal contact, which raises some puzzling question. How did the Universe "know" to be at the same temperature over such enormous scales (like today's horizon size) if only such small pieces were connected?

3 (e)

(10 pts) In matter dominated phase of a flat Universe, find how a density fluctuation $\delta\rho/\rho$ grows with z. Baryonic fluctuations, which are what is visible in the CMB, were at the time of recombination of order 10^{-4}. Would these fluctations alone had time to grow to the point of collapse ($\delta\rho/\rho = 1$) by the present day?

If we have two regions of the same Universe, the background flat with density ρ and the background with a higher density ρ' and consequently positive curvature, then the two expansion equations are

$$
\begin{aligned}
H^2R^2 - \frac{8}{3}\pi G\rho R^2 &= 0 \\
H^2R^2 - \frac{8}{3}\pi G\rho' R^2 &= -k'c^2 \\
\frac{8}{3}\pi G R^2\,(\rho' - \rho) &= k'c^2 \\
(\rho' - \rho) &= \frac{3k'c^2}{8\pi G R^2} \\
\frac{\rho' - \rho}{\rho} &= \frac{3k'c^2}{8\pi\rho G R^2} \\
\frac{\delta\rho}{\rho} &= \frac{3k'c^2}{8\pi\rho G R^2} \\
&= \frac{3k'c^2}{8\pi\rho_0 R^{-3} G R^2} \\
&= \frac{3k'c^2}{8\pi\rho_0 G} R
\end{aligned}
$$

so the density fluctuation grows as R, or as $(1+z)^{-1}$. Thus, if a perturbation had amplitude 10^{-4} at a redshift $z = 1100$, then the amplitude at the current time would be 1101 times larger, or $(\delta\rho/\rho)_0 = 1.101 \times 10^{-1}$ – so well too small to collapse at the present day.