AST 222 Winter 2011

Assignment #2

Due 12pm, Mon Feb 14

This assignment is due before class on Feb 14; that is, at 12:10 **sharp**. Assignments handed in late — including at the end of class — will lose 20% credit. Answers should be posted online no later than Tuesday; once answers are posted, of course, no further assignments will be accepted. We'll have assignments handed back promptly.

The instructor will have office hours 1-1:30pm Mon and Wed; the TAs have office hours 3-4pm Tues and 4-5pm Thurs. Questions can also be asked by email to `ast222@astro.utoronto.ca`. "Online office hours" will be held in the Blackboard AST222 chat at 7pm on Sunday, Feb 13.

Show your work, and good luck!

Question 1 - Short Answer [15 pts]

None of these should require more than a few sentences or lines of math.

1 (a)

(2 pts) Describe, in a few words and a diagram, the ELS (Eggen, Lynden-Bell, & Sandage) model of galaxy formation.

In the ELS model, the galaxy forms form the collapse of a large cloud of gas with a small amount of net initial rotation. As the collapse continues, the proto-galaxy spins up. Early star formation produces the halo; when the final "pancake-like" collapse occurs, the disk forms.

1 (b)

(2 pts) Why does this not accurately describe the formation of our Milky Way?

Because such a collapse is actually quite fast, it would produce a halo much younger than we see, with a much smaller range of ages; similarly, the spread of ages and metalicities in the disk is wrong.

1 (c)

(4 pts) Name two things we'd expect to see a lot more of in a spiral galaxy than an elliptical galaxy, and two things we'd expect to see a lot more of in an elliptical galaxy than a spiral.

In spiral galaxy, we expect to see much more young stars and gas; other answers could include magnetic fields, spiral structure, disk structure. In an elliptical one would expect to see more globular clusters, more stars generally (they're larger), and more old stars in particular.

1 (d)

(4 pts) Give two pieces of evidence (or supporting arguments) used at the time for each side of the Curtis-Shapely debate.

Curtis: (pro small-galaxy sizes)

- Observed M101 rotation incosistent with large galactic sizes
- M31 Novae inconsistant with large galactic sizes

Shapley: (pro large-galaxy sizes)

- Nova still require M31 to be very distant
- Spiral nebulae show no proper motion
- MW looking like other spiral nebulae explains zone of avoidance

1 (e)

(3 pts) What is the Tully-Fisher relation, and describe two things we can learn from it.

The Tully-Fisher relation relates the luminosity of a spiral galaxy and its maximum rotation speed; roughly, the luminosity scales as the maximum rotation velocity to the fourth power. From this we can obtain both a distance measure (by inferring the actual luminosity and measuring the observed flux, we can determine a distance by inverse-square law), and we have a constraint on galaxy formation theories (they must reproduce this observed relation).

Question 2 - Collisional Relaxation and Omega Centauri[30 pts]

Omega Centauri is a massive Milky Way globular cluster, with mass $M \approx 5 \times 10^6\, M_\odot$ and a radius of approximately $26\,\mathrm{pc}$. Because it is is composed of a population of very old (like 12 billion years old) stars, the average mass of a star within ω Cen is smaller than in the disk, say about $0.5\, M_\odot$.

In this question, we'll look at the collisional relaxation time of the cluster, and consider its interaction with the disk. We'll end up finding out why the disk manages to maintain such a low velocity dispersion - even interactions with "hot" halo populations happen too infrequently or slowly to impart significant velocity dispersion to the youngest disk stars. The picture here is that the main way for the cluster to drive velocity dispersions in the disk is when the two actually overlap, and the disk stars, which normally have a very small dispersion, "feel" the velocity dispersion of the cluster, start to become collisionally relaxed by it. (There's another possible effect; that simply the gravitational perturbation from the cluster in the disk will scatter disk stars. Because this happens slowly, in class we'll argue it leads to only a small net increase in dispersion). Assume that the inclination of Omega Centauri's orbit with respect to the plane is 15 degrees; that is, it grazes the disk at a fairly low angle rather than plunging in from near the poles.

2 (a)

(3 pts) An isolated system such as this can almost certainly be assumed to be in virial equillibrium. What should the velocity dispersion in the cluster be?

The virial theorem tells us

$$2K = -U$$

or, for a spherical system that we can approximate as uniform density

$$< v^2 > \quad = \quad \frac{3}{5}\frac{GM}{R}$$

$$\begin{aligned} < v^2 > &= \frac{6}{5}\frac{4.3\times10^{-3}\mathrm{pc}\,(\mathrm{km\,s^{-1}})^2 M_\odot^{-1}\times 5\times10^6\,M_\odot}{26\mathrm{pc}} \\ &= 496.3\,(\mathrm{km\,s^{-1}})^2 \\ \sigma &= 22.3\,\mathrm{km\,s^{-1}} \end{aligned}$$

2 (b)

(2 pts) Recent observations of proper motions within the cluster have found suggestions of a surprisingly large net rotation, of about $7.9\,\mathrm{km\,s^{-1}}$. Is this larger or smaller than the velocity dispersion? Would you expect this system to be more disk like or more spherical?

Although the two velocities are comparable, the velocity dispersion is clearly larger. Compare to the disk of our galaxy, where the velocity dispersion is 1/10th or less of the rotation speed. So it should be more spherical than disk-like.

2 (c)

(3 pts) Virialization is quite fast; distributing the velocities isotropically is a much slower process. What is the collisional relaxation time for this cluster?

The collision time is given by

$$t_{\mathrm{col}} = \frac{R}{v}\frac{N}{\log N}$$

where R is the radius, v is the typical velocity (here we can use the velocity dispersion), and N is the number of objects. If the mass of the system is $5\times10^6\,M_\odot$ and the average mass of a star in the cluster is $0.5\,M_\odot$, then the number of stars is 10^7, and we have

$$\begin{aligned} t_{\mathrm{col}} &= \frac{R}{v}\frac{N}{\log N} \\ &= \frac{26\,\mathrm{pc}}{22.3\,\mathrm{km\,s^{-1}}}\frac{10^7}{\log 10^7} \\ &= 1.14\times10^6\,\mathrm{yr}\times1.43\times10^6 \\ &= 1.6\times10^{12}\,\mathrm{yr} \end{aligned}$$

which is over a thousand billion years, well in excess of the age of the Galaxy (or the Universe).

2 (d)

(1 pt) The globular cluster is located approximately $6.8\,\mathrm{kpc}$ from the Galactic centre. If the rotation curve is flat between there and here, what is the speed of the clusters orbit around the galactic centre? (We'll assume circular orbit).

A flat rotation curve means constant rotation speed; so the rotation speed would be the same as at the Sun's position, or $220\,\mathrm{km\,s^{-1}}$.

2 (e)

(6 pts) The clusters orbit necessarily takes the cluster through the plane of the disk. When this occurs, the cluster stars pass through the disk, and the disk stars "see" a new local population of stars with much higher velocity dispersion than the disk stars.

For how long does the globular cluster intersect the midplane of the disk? To what fraction of a collisional relaxation time does this correspond? In our calculation in class, the growth in velocity dispersion σ due to collisional relaxation was linear in time; what fraction of the velocity dispersion of the cluster is imparted to disk stars in one crossing?

The globular cluster is of radius $26\,\mathrm{pc}$ (and thus diameter of $52\,\mathrm{pc}$) travelling at $220\,\mathrm{km\,s^{-1}}$, so that it will intersect the midplane for a length of time $t = D/(v \sin i) = 8.9 \times 10^5\,\mathrm{yr}$. Since the growth in velocity dispersion is linear in time, and this is a length of time that is $8.9 \times 10^5/1.6 \times 10^{12} = 5.5 \times 10^{-7}$ of the collisional relaxation time, we expect that the velocity dispersion imparted will be that fraction of the internal velocity dispersion, or $1.2 \times 10^{-5}\,\mathrm{km\,s^{-1}}$ – in other words, tiny.

[Note that you've been asked above to use the *internal* collisional relaxation time of the cluster to estimate the relaxation time of the disk stars encountering the cluster. This is an ok estimate, but a more careful calculation would use the relative velocity of the disk stars relative to the cluster. A little trig shows that this velocity is $57.4\,\mathrm{km\,s^{-1}}$ so the proper collisional relaxation time would be

$$t_{\mathrm{col}} = \frac{R_{\mathrm{cluster}}}{57.4\,\mathrm{km\,s^{-1}}} N \log N$$

and compare that to the crossing time. Numerically, this is a fairly big change - about $2.5\times$ — and the velocity dispersion changes now to $3.1 \times 10^{-5}\,\mathrm{km\,s^{-1}}$ — but the answer doesn't change much; this is still tiny compared to observed dispersions.]

2 (f)

(6 pts) There are of order 150 known globular clusters in our galaxy, of varying sizes. Assuming that the masses of the individual stars that make them up are similar (they're all old stellar populations), and ignoring logarithmic terms (which only vary by a factor of a few over factors of 1000 in size of the cluster), argue that rates of velocity dispersion deposition (σ/t_{col}) depend only on the density (or, more specifically, their radius R).

The collisional relaxiation time is

$$t_{\mathrm{col}} = \frac{R}{v} \frac{N}{\log N}$$

and the relevant v for the velocity is the velocity dispersion $\sigma = \sqrt{< v^2 >}$:

$$\sigma = \sqrt{\frac{3}{5} \frac{GM}{R}}.$$

So the growth in velocity dispersion per unit time is

$$\dot{\sigma} = \frac{\sigma}{t_{\mathrm{col}}} = \frac{\sigma}{\frac{R}{\sigma} \frac{N}{\log N}} = \frac{\sigma^2 \log N}{RN}$$

If they are all made of similar objects of mass m, then $M = Nm$ and we have

$$\dot{\sigma} = \frac{3GNm \log N}{5R^2 N} = \frac{6Gm \log N}{5} \frac{1}{R^2}.$$

Note too that the amount of velocity dispersion deposition per collision is also proportional to R, making the total proportional only to $1/R$.

2 (g)

(3 pts) How often does Omega Centauri cross the plane? If 150 clusters are imparting similar velocity dispersions similarly often, is this enough to increase the velocity disperison above say 10% of the rotation

speed frequently seen by middle-aged disk stars like our sun (which has a motion with respect to the local standard of rest of about $16.5\,\mathrm{km\,s^{-1}}$?) over the past billion years?

Omega Centauri crosses the plane twice every period, $2\pi 6.8\,\mathrm{kpc}/220\,\mathrm{km\,s^{-1}} = 190.\,\mathrm{Myr}$, or approximately 10 times in the last billion years. Multiplying this by 150 gives us 1700 interactions, each presumably imparting $1.2\times10^{-5}\,\mathrm{km\,s^{-1}}$ in velocity dispersion, leaving us only about $0.021\,\mathrm{km\,s^{-1}}$ of velocity dispersion - not enough.

In addition, we've neglected the fact that these interactions are very local; each intersection of Omega Centauri with the disk only effects an area of the disk the size of Omega Centauri.

2 (h)

(6 pts) The stellar halo, consisting of similarly old stars, has a mass of approximately $1\times10^{9}\,M_\odot$ and extends out to a radius of approximately $20\,\mathrm{kpc}$. What is the expected velocity dispersion and collisional relaxation time of this system? The disk is completely encased in this system; is collisional relaxation of the disk stars by the halo enough to have increased the velocity dispersion of the disk to beyond 10% levels in the past billion years?

The velocity dispersion of the system is

$$\sigma = \sqrt{3GM}5R = 8.1\,\mathrm{km\,s^{-1}}.$$

The collision time of the system is

$$t_{\mathrm{col}} = \frac{R}{v}\frac{N}{\log N} = 5.4\times10^{17}\,\mathrm{yr}.$$

While this does effect the entire disk at once, it is so enormously slow that it is a factor of 8×10^{8} off bringing up the velocity dispersion to the observed levels.

[Again, this is the *self*-collisional relaxation time; the disk stars are actually moving at the full $220\,\mathrm{km\,s^{-1}}$ relative to the halo, so a more carefully calculated collisional relaxation time would be

$$t_{\mathrm{col}} = \frac{20\,\mathrm{kpc}}{220\,\mathrm{km\,s^{-1}}}\frac{2\times10^{9}}{\log 2\times10^{9}}$$

which is a huge numerical difference (27×!) but the resulting time only comes down to $2.0\times10^{16}\,\mathrm{yr}$ – still a million times longer than the age of the Universe.]

Question 3 - Epicycles in Power-Law Potentials [40 pts]

The instructors of AST222 and some physics courses are meeting in their space-station lair within the nearby Sinister Nebulae, congratulating each other on having arranged for all of their assignments to have been due the same day and scheming as to how to do it again while still having it appear accidental. Space Lair Instructo is orbiting the centre of the nebulae — a small stellar remnant, as dead as their shrivelled hearts — and the nebula itself is a spherical gas cloud. They are orbiting at a distance of $1\,\mathrm{AU}$, and the mass interior to their orbit is $1\,M_\odot$.

We'll be considering the epicyclic motion of the Lair. In dealing with these more complicated gravitational dynamics problems, we use potentials quite a bit. They are useful quantities, because once you know the potential due to gravity Φ_g, you know almost everything, simply by taking deriviatives; for instance

$$\begin{aligned} -\nabla\Phi &= \vec{g} \\ \nabla^2\Phi &= 4\pi G\rho \end{aligned}$$

but it's much easier to take derivatives than do integrals, and Φ also has the advantage of being a scalar.

Since we will be assuming spherical symmetry, this simplifies quite a bit:

$$\begin{aligned} -\nabla\Phi_g &= \vec{g} \\ -\frac{d}{dr}\Phi_g\hat{r} &= g\hat{r} \\ \frac{d}{dr}\Phi_g &= \frac{GM(<r)}{r^2} \end{aligned}$$

We'll further be interested in the effective potential,

$$\frac{d}{dr}\Phi_{\text{eff}} = \frac{d}{dr}\left(\Phi_g + \Phi_c\right) = \frac{GM(<r)}{r^2} + \frac{d}{dr}\frac{J_z^2}{2r^2}$$

where J_z is a constant, the specific angular momentum of the object in question, $J_z = r_0^2\Omega_0$.

3 (a)

(4 pts) We don't yet know the distribution of mass in the Sinister nebula. Find the epicyclic frequency $\kappa = -\frac{d^2}{dr^2}\Phi_{\text{eff}}$ in terms of the rotation curve $V(R)$ and its deriviatives for a density distribution that results in an arbitrary rotation curve $V = V(R)$. You may find it useful to write

$$\kappa^2 = \frac{d}{dr}\left[\frac{d}{dr}\Phi_g + \frac{d}{dr}\Phi_c\right]\Bigg|_{r_0},$$

express this in terms of velocities, simplify, take the final derivative, and evaluate at some position R_0.

We'll start at the suggested starting point:

$$\begin{aligned} \kappa^2 &= \frac{d}{dr}\left[\frac{d}{dr}\Phi_g + \frac{d}{dr}\Phi_c\right]\Bigg|_{r_0} \\ &= \frac{d}{dr}\left[\frac{GM(<r)}{r^2} + \frac{d}{dr}\frac{J_z^2}{2r^2}\right]\Bigg|_{r_0} \\ &= \frac{d}{dr}\left[\frac{V^2(r)}{r} + \frac{d}{dr}\frac{r_0^2V_0^2}{2r^2}\right]\Bigg|_{r_0} \\ &= \frac{d}{dr}\left[\frac{V^2(r)}{r} - \frac{r_0^2V_0^2}{r^3}\right]\Bigg|_{r_0} \\ &= \left[\frac{2V(r)V'(r)r - V^2(r)}{r^2} + \frac{3r_0^2V_0^2}{r^4}\right]\Bigg|_{r_0} \\ &= \left[\frac{2V_0V'(r_0)}{r_0} - \frac{V_0^2}{r_0^2} + \frac{3V_0^2}{r_0^2}\right] \\ &= 2\left[\frac{V_0V'(r_0)}{r_0} + \frac{V_0^2}{r_0^2}\right] \\ &= 2\left[\frac{V_0}{r_0}\frac{dV}{dr}\bigg|_{r_0} + \frac{V_0^2}{r_0^2}\right] \\ &= 2\frac{V_0}{r_0}\left[\frac{dV}{dr}\bigg|_{r_0} + \frac{V_0}{r_0}\right] \end{aligned}$$

3 (b)

(4 pts)

Using the result above, prove the equation that was shown in class relating the epicyclic frequency, orbital frequency, and Oort constants, giving the number of epicycles per orbit as a function of locally-measurable parameters:

$$\frac{\kappa_0}{\Omega_0} = 2\left(\frac{-B}{A-B}\right)^{1/2}.$$

We recall the Oort constants:

$$\begin{aligned} A &= -\frac{1}{2}\left(\left.\frac{dV}{dR}\right|_{R_0} - \frac{V_0}{R_0}\right) \\ B &= -\frac{1}{2}\left(\left.\frac{dV}{dR}\right|_{R_0} + \frac{V_0}{R_0}\right) \end{aligned}$$

and that $\Omega_0 = V_0/r_0$. Ω_0 is clearly equal to $(A-B)$; κ_0 is clearly very closely related to B:

$$\begin{aligned} \kappa_0^2 &= -2\frac{V_0}{r_0}\left[\left.\frac{dV}{dr}\right|_{r_0} + \frac{V_0}{r_0}\right] \\ &= -4\Omega_0 B \\ \kappa_0 &= 2\left(-\Omega_0 B\right)^{1/2} \\ \frac{\kappa_0}{\Omega_0} &= 2\frac{\left(-\Omega_0 B\right)^{1/2}}{\Omega_0} \\ &= 2\left(\frac{-B}{\Omega_0}\right)^{1/2} \\ &= 2\left(\frac{-B}{A-B}\right)^{1/2} \end{aligned}$$

3 (c)

(5 pts) Now consider power-law rotation curves as in the first assignment. Take your expression for part (a) and simplify for this particular case,

$$V(R) = V_0\left(\frac{R}{R_0}\right)^p.$$

Given the data (R_0 and $M(< R_0)$) above, numerically evaluate the epicyclic frequency for Keplerian ($p = -1/2$), flat ($p = 0$), and solid-body rotation ($p = 1$).

We start with

$$\kappa^2 = 2\frac{V_0}{r_0}\left[\left.\frac{dV}{dr}\right|_{r_0} + \frac{V_0}{r_0}\right]$$

which, when applying this particular functional form,

$$\begin{aligned} \frac{dV}{dR} &= p\frac{V_0}{R_0}\left(\frac{R}{R_0}\right)^{p-1} \\ &= p\frac{V(R)}{R} \end{aligned}$$

$$\begin{aligned} \kappa^2 &= 2\frac{V_0}{r_0}\left[p\frac{V_0}{r_0} + \frac{V_0}{r_0}\right] \\ &= 2(p+1)\left(\frac{V_0}{r_0}\right)^2 \\ \kappa &= \sqrt{2(p+1)}\frac{V_0}{r_0} \end{aligned}$$

Note that, for $p = -1/2$ (Keplarian rotation), $\kappa = V_0/r_0 = \Omega_0$.

We know $V_0 = \sqrt{GM/R} = 29.8\,\mathrm{km\,s^{-1}}$ and $R_0 = 1\,\mathrm{AU}$, so $\Omega_0 = 2.0 \times 10^{-7}\,\mathrm{s^{-1}}$. for the required p, then, we have

p	κ
$-1/2$	1.99×10^{-7} sec^{-1}
0	2.81×10^{-7} sec^{-1}
1	3.98×10^{-7} sec^{-1}

We've now calculated epicyclic frequencies, but not the epicycles themselves. All small perturbations of a circular orbit generate epicyclic motion with the same frequency, but the size - in displacement and velocity - of the epicycle depends on the size of the perturbation.

While the instructors were having a spirited discussion of what midterm test format was the best for inspiring fear and agony, no one was watching the helm and a head-on collision occurred with a stray piece of space rock. The collision was quite strong; the Lair lost 1% of its forward momentum. Measurement of the Oort constants by the instructors had previously determined that the orbit was such that the rotation curve within the nebula was flat.

3 (d)

(6 pts) What was the new equilibrium radius of the Lair's orbit; that is, at what radius would the new angular momentum result in a circular orbit?

We need to find the r so that the new angular momentum $m(0.99v_0)r_0$ corresponds to the angular momentum of circular rotation at that orbit. Since the rotation curve is flat, we know the new rotation speed is still V_0, so that the new position must be $0.99r_0$, or $0.99\,\mathrm{AU}$.

3 (e)

(6 pts) The Lair will fall down to this new equilibrium radius; as it does so, it will gain radial velocity, with a kinetic energy gain equal to the potential energy loss as it falls inwards. At the new equilibrum radius, what is the radial velocity perturbation?

As the lair falls inwards, work is being done upon it

$$\Delta U = \int_r^{r'} F dr = \int_r^{r'} \frac{GM(<r)m}{r^2} dr.$$

To evalate this, we need to know how $M(<r)$ varies with r. For a flat rotation curve, $GM(<r)/r$ remains constant, so $M(<r) \propto r$; in particular, for this case $M(<r) = 1\,M_\odot(r/(1\,\mathrm{AU}))$. Thus

$$\Delta U = \int_r^{r'} \frac{GM(<r)m}{r^2} dr$$

$$
\begin{aligned}
&= \int_r^{r'} \frac{G(1\,M_\odot)m}{r(1\,\mathrm{AU})} dr \\
&= \frac{G(1\,M_\odot)m}{1\,\mathrm{AU}} \int_r^{r'} \frac{dr}{r} \\
&= (29.8\,\mathrm{km\,s^{-1}})^2 m \left[\ln r\right]_{r=1\,\mathrm{AU}}^{r=0.99\,\mathrm{AU}} \\
&= (29.8\,\mathrm{km\,s^{-1}})^2 m \ln \frac{1\,\mathrm{AU}}{0.99\,\mathrm{AU}} \\
&= 8.93 m (\mathrm{km\,s^{-1}})^2.
\end{aligned}
$$

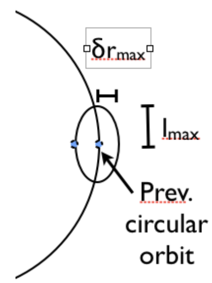
this must be converted into a corresponding kinetic energy,

$$
\begin{aligned}
\frac{1}{2} m v'^2 &= 8.93 m (\mathrm{km\,s^{-1}})^2 \\
v' &= 4.22\,\mathrm{km\,s^{-1}}
\end{aligned}
$$

3 (f)

(5 pts)
Now this forward-motion perturbation looks just like a radial-velocity perturbation at the new equilibrium radius. We expect the radial motions to look like a simple harmonic oscillator:

$$
\begin{aligned}
\delta v_r &= -\delta v_{r,\mathrm{max}} \sin(2\pi\kappa t) \\
\delta r &= \delta r_{\mathrm{max}} \cos(2\pi\kappa t)
\end{aligned}
$$



with $V_{r,\mathrm{max}}$ being the radial velocity we've just calculated above. Relate $\delta_{r,\mathrm{max}}$ and $V_{r,\mathrm{max}}$ and find the radial extent of the epicyclic motion that the Lair undergoes.

The velocity, of course, has to be the time deriviative of the displacement:

$$
\begin{aligned}
\dot{\delta r} &= -2\pi\kappa\delta r_{\mathrm{max}} \sin(2\pi\kappa t) \\
\delta v_r &= -2\pi\kappa\delta r_{\mathrm{max}} \sin(2\pi\kappa t) \\
-\delta v_{r,\mathrm{max}} \sin(2\pi\kappa t) &= -2\pi\kappa\delta r_{\mathrm{max}} \sin(2\pi\kappa t) \\
\delta v_{r,\mathrm{max}} &= 2\pi\kappa\delta r_{\mathrm{max}} \\
\delta r_{\mathrm{max}} &= \frac{\delta v_{r,\mathrm{max}}}{2\pi\kappa}
\end{aligned}
$$

Now note that κ actually varies with position in the disk — in particular, from above we have that $\kappa_0 = \sqrt{2(p+1)}V_0/r_0$ or $\kappa = \kappa_0(r_0/r)$. So at the new position, $\kappa = 2.84 \times 10^{-7}\ \mathrm{sec}^{-1}$, and

$$
\begin{aligned}
\delta r_{\mathrm{max}} &= \frac{4.22\,\mathrm{km\,s^{-1}}}{2\pi(2.81 \times 10^{-7}\ \mathrm{sec}^{-1})} \\
&= 2.39 \times 10^9\ \mathrm{m} = 2.39 \times 10^6\ \mathrm{km}
\end{aligned}
$$

this works out to $0.016\,\mathrm{AU}$, similar to the $0.01\,\mathrm{AU}$ which the lair originaly fell.

3 (g)

(10 pts) We can do the same with the epicyclic motion along the direction of the orbit. Using the conservation of angular momentum along the epicycle, find the maximum velocity deviation in the direction of orbital rotation on the innermost part of the epicycle, and use the same simple harmonic motion calculation

$$\begin{aligned} \delta V &= -\delta V_{\max} \sin(2\pi\kappa t) \\ \delta l &= l_{\max} \cos(2\pi\kappa t) \end{aligned}$$

to find out $l_{\max}$, the maximum azimuthal deviation of the epicycle from circular orbit.

By conservation of angular momentum,

$$\begin{aligned} m(V + \delta V_{\max})(r - \delta_{r,\max}) &= mVr \\ (V + \delta V_{\max}) &= \frac{Vr}{r - \delta_{r,\max}} \\ \delta V_{\max} &= V\left(\frac{r}{r - \delta_{r,\max}} - 1\right) \\ \delta V_{\max} &= V\left(\frac{1\,\text{AU}}{1\,\text{AU} - 0.016\,\text{AU}} - 1\right) \\ \delta V_{\max} &= 0.0162V \\ &= 0.48\,\text{km}\,\text{s}^{-1} \end{aligned}$$

and by the same argument as above,

$$\begin{aligned} 2\pi\kappa l_{\max} &= \delta V_{\max} \\ l_{\max} &= \frac{\delta V_{\max}}{2\pi\kappa} \\ &= 2.68 \times 10^8\,\text{m} = 2.68 \times 10^5\,\text{km} \end{aligned}$$

so in this case, the epicycle is elongated in the radial direction.

Question 4 - Central SMBHs [15 pts]

Table 1 shows approximate data from Bender *et al.*, (2005) looking at spectra from nearby galaxy M31, the Andromeda galaxy. The tables represent velocity dispersion and rotation curves very near the centre of the galaxy. Graphs of this data are shown in the Lecture notes for Lecture 9. Data files containing this data will be available on the problem sets webpage in various formats.

4 (a)

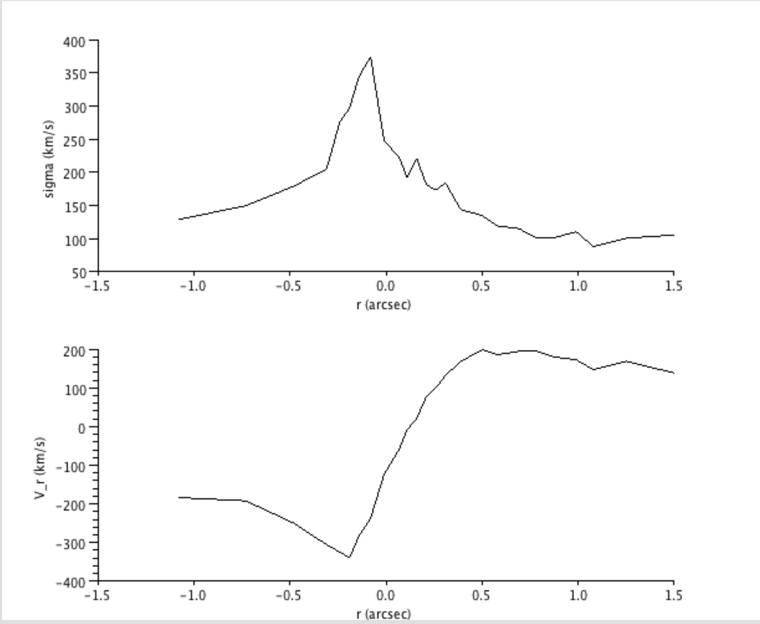
(2 pts) Plot the data, using your favourite piece of software. M31 is approximately $2.5\,\text{Mpc}$ away. To what length does 0.1 arcsec correspond?

0.1 arcseconds is 4.85×10^{-7} radians; so at a distance of $2.5\,\text{Mpc}$, 0.1 arcseconds corresponds to $2.5\,\text{Mpc} \times 4.85 \times 10^{-7} = 1.212\,\text{pc}$.

A plot of the data should look something like shown below.

R (arcsec)	$\sigma\,(\mathrm{km\,s^{-1}})$	$V_{\mathrm{rot}}\,\mathrm{km\,s^{-1}}$
-1.08	128.8	-183.5
-0.73	149.7	-192.1
-0.48	179.1	-249.7
-0.31	204.9	-305.2
-0.24	276.1	-324.4
-0.19	296.9	-339.3
-0.14	344.8	-281.7
-0.08	374.2	-236.9
-0.01	247.9	-123.8
0.07	222.1	-57.6
0.11	191.4	-8.5
0.16	220.9	21.3
0.21	181.6	76.8
0.26	173.0	100.3
0.31	184.0	132.3
0.39	143.6	168.6
0.50	135.0	198.5
0.58	119.0	185.7
0.69	115.3	194.2
0.79	100.6	194.2
0.88	101.8	179.3
0.99	110.4	172.9
1.08	88.3	147.3
1.25	100.6	168.6
1.50	105.5	138.7

Table 1: Angular offset from inferred centre of galaxy, R, in arcseconds; measured velocity dispersion in $\mathrm{km\,s^{-1}}$; and measured rotation speed, in $\mathrm{km\,s^{-1}}$.



4 (b)

(7 pts) When looking at a spectra, how does one infer a velocity dispersion? A rotation velocity? Consider the rotation curve jump within the inner ± 0.5 arcsec. Estimate a mass enclosed in the region necessary to produce those velocities.

When observing a spectrum at a point, a velocity dispersion can be measured by looking at the line width, with is spread out due to the range of velocities occuring within the single line of sight. Rotation velocities, on the other hand, can only be measured by looking at spectra across the object, and finding the variation of the mean velocity across the body.

The rotation curve jump in the inner regions is complicated by the feature at about -0.2 arcseconds; but the overall jump in rotation speed is pretty clearly approximately $200\,\mathrm{km\,s^{-1}}$, over the length of $6.06\,\mathrm{pc}$, or

$$V^2 = \frac{GM}{r}$$
$$M = \frac{rV^2}{G} = 5.7 \times 10^6\, M_\odot$$

4 (c)

(6 pts) Consider the peak in velocity dispersion in same region. Estimate the mass enclosed in the region necessary to give such a dispersion (assume spherical distribution). Is this consistent, to within an order of magnitude, of your answer above?

Looking at the whole region under consideration, the dispersion increases $220\,\mathrm{km\,s^{-1}}$ over the same $6.06\,\mathrm{pc}$, giving us

$$\sigma^2 = \frac{3}{5}\frac{GM}{r}$$
$$M = \frac{5rsigma^2}{3G} = 1.1 \times 10^7\, M_\odot$$

or about a factor of two larger than the rotation curve estimate.

Interestingly, though, with velocity dispersion it seems that we can localize this feature even more markedly; here there is clearly a velocity displersion jump of $170\,\mathrm{km\,s^{-1}}$ over a gap of just ± 0.19 arcseconds, leading to

$$\begin{aligned} \sigma^2 &= \frac{3}{5}\frac{GM}{r} \\ M &= \frac{5r\sigma^2}{3G} = 2.6 \times 10^6\,M_\odot \end{aligned}$$

that is, the bound is a somewhat lower mass, but in a much smaller (6%) volume.

Note that this is a direct dynamical measurement of the central mass. Another way to go, pointed out by a couple very attentive students, is to use the observed scaling between central SMBH mass and bulge velocity dispersion, from (say) lecture 9:

$$\begin{aligned} M_{\mathrm{BH}} &= 1.35 \times 10^8\,M_\odot \left(\frac{\sigma_e}{200\,\mathrm{km\,s^{-1}}}\right)^4 \\ &= 1.35 \times 10^8\,M_\odot \left(\frac{170\,\mathrm{km\,s^{-1}}}{200\,\mathrm{km\,s^{-1}}}\right)^4 \\ &= 7.1 \times 10^7\,M_\odot. \end{aligned}$$

Now, this isn't quite right, because this observed scaling is for the relationship between SMBH mass and the velocity dispersion in the *entire* bulge, rather than just for a localized region of it; so a better σ to use would be more like $100\,\mathrm{km\,s^{-1}}$.

Note that this is somewhat less accurate when we have more detailed information available (as in this case, where we can see a very sharply peaked dispersion). In addition, this is a somewhat odd case - we can see that where we get a much higher mass than expected. Another way of thinking of this is that this lower mass seems to be much more effective at generating dispersion than usual; this is because this somewhat odd galaxy seems to have not one but *multiple* SMBHs at its centre, orbiting each other and inducing significantly higher velocity dispersions than would normally be the case.