Cosmological Black Holes as Models of Cosmological Inhomogeneities

by

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Abstract

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Since cosmological black holes modify the density and pressure of the surrounding universe, and introduce heat conduction, they produce simple models of cosmological inhomogeneities that can be used to study the effect of inhomogeneities on the universe's expansion. In this thesis, new cosmological black hole solutions are obtained by generalizing the expanding Kerr-Schild cosmological black holes to obtain the charged case, by performing a Kerr-Schild transformation of the Einstein-de Sitter universe (instead of a closed universe) to obtain non-expanding Kerr-Schild cosmological black holes in asymptotically-flat universes, and by performing a conformal transformation on isotropic black hole spacetimes to obtain isotropic cosmological black hole spacetimes. The latter approach is found to produce cosmological black holes with energy-momentum tensors that are physical throughout spacetime, unlike previous solutions for cosmological black holes, which violate the energy conditions in some region of spacetime. In addition, it is demonstrated that radiation-dominated and matter-dominated Einstein-de Sitter universes can be directly matched across a hypersurface of constant time, and this is used to generate the first solutions for primordial black holes that evolve from being in radiation-dominated background universes to matter-dominated background universes. Finally, the Weyl curvature, volume expansion, velocity field, shear, and acceleration are calculated for the cos-

mological black holes. Since the non-isotropic black holes introduce shear, according to Raychaudhuri's equation they will tend to decrease the volume expansion of the universe. Unlike several studies that have suggested the relativistic backreaction of inhomogeneities would lead to an accelerating expansion of the universe, it is concluded that shear should be the most likely influence of inhomogeneities, so they should most likely decrease the universe's expansion.

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Contents

Chapter 1

Introduction

General Relativity is necessary to understand systems once the mass M becomes of the order of the spatial size S, so it is needed on small scales for understanding highly condensed objects, but it is also needed on large scales for understanding systems as diffuse as the universe, since with $M \sim S^3$ even the smallest mass density will be significant on large enough scales. Thus, cosmological black holes unite two relativistic extremes together in one model. Cosmological black holes are of interest either as examples of non-isolated, time-dependent black hole solutions or as inhomogeneous cosmological models that allow for the study of the influence of inhomogeneities on the universe.

The study of cosmological inhomogeneities is important because the universe isn't homogeneous on all scales, yet models of completely homogeneous universes are commonly used to represent it, either with the expectation that the general evolution of a universe with inhomogeneities will in no way differ from that of a completely smooth universe, or with the belief that the expansion of the universe is governed by a perfectly homogeneous spatial expansion regardless of local inhomogeneities in the matter density. Considering that gravitational entropy is maximized when matter clumps, it is actually surprising that the universe started off as close to homogeneous as it did, and it would be even more surprising if the universe weren't becoming more inhomogeneous with time, yet many people seem to expect the universe should be perfectly homogeneous. In order to explain the homogeneity of the early universe, scenarios such as inflation have been hypothesized, but there is no proof that inflation actually happened, so the homogeneity of the early universe remains unexpected. Bildhauer & Futamase (1991) have speculated that the relativistic backreaction of inhomogeneities on the background universe may mimic the effect of a cosmological constant, and Zehavi et al. (1998) have suggested that the cosmological constant may merely be an artifact of living in an underdense region of the universe such that we observe the universe to be expanding more slowly as we look further away and equivalently further back in time. Thus, it is necessary to account for the possible influence of inhomogeneities if we are to properly understand the global nature of the universe.

In this chapter, the nature of spacetime will be discussed to serve as a backdrop for this thesis, and then the motivation for studying inhomogeneities will be discussed in further detail. The specific background knowledge and previous work necessary to understanding the work in this thesis will appear in Chapter 2, and then new cosmological black hole solutions will be detailed in Chapters 3 and 4. In Chapter 5 it will be shown that radiation-dominated universes can be matched to matter-dominated universes across hypersurfaces of constant time, and this will be used to obtain solutions for primordial cosmological black holes that start off in the radiation-dominated phase of the universe. Finally, the effect of cosmological black holes on the expansion of the universe will be examined in Chapter 6.

The sign conventions used in this thesis are signature $(- + + +)$ and negative Einstein sign ($G_{ab} = -\kappa T_{ab}$). Geometrized units ($G = c = 1$) are generally used (thus $\kappa = 8\pi$). The notation $u_{|a}$ denotes partial differentiation of u, and the notation $u_{||a}$ denotes covariant differentiation.

1.1 The Nature of Spacetime

Commonly people speak of the expansion of the universe by talking about space as though it is a rubber sheet that expands, carrying the galaxies away from one another, although we know no ether or anything akin to a rubber sheet exists, so we know we can never observe absolute positions and velocities (regardless of whether any sort of absolute space even exists). Empirically then, it only makes sense to consider the velocity field of the matter and how everything is moving relative to everything else, rather than considering the expansion of space. The redshifting of light is often described in terms of photons getting stretched by the expansion of space; however, we know from Special Relativity that photons don't have their own rest frames, so they don't have an intrinsic wavelength that could get stretched by the expansion of space. Thus, it is really only appropriate to speak of the photons as getting redshifted relative to the matter: relative to the matter they are emitted from, they will always have the same wavelength in that frame of reference, and relative to the matter they are absorbed by, they would have always had the same wavelength in that frame of reference.

In homogeneous cosmology, spacetime is foliated by a series of homogeneous spatial slices that each exist at a given time such that space and time appear to be uniquely specified, although we know from Special Relativity that space and time aren't truly separate entities and there isn't really a unique way to slice up spacetime such that every observer agrees on the surfaces of simultaneity. It is only because we deal with spatially-homogeneous models (as opposed to spatially-inhomogeneous or spacetime-homogeneous models) that there are different surfaces of constant density that every observer in the universe can decide to consider as the surfaces of simultaneity. If the observers don't move relative to the universe's matter, then their surfaces of simultaneity will simply be the homogeneous spatial slices, so it will be natural to see the universe as being spatially homogeneous. However, there is nothing truly special about this foliation of spacetime that makes the homogeneous spatial slices the true surfaces of simultaneity or requires that spacetime be foliated into homogeneous spatial slices. If an observer is boosted to move relative to the background distribution of matter in the universe, then that observer will see the surfaces of simultaneity as being inhomogeneous, with an asymmetry in density along the direction of motion. Thus, it is just a convenience to assume that all observers will agree on what is space and what is time in a spatially homogeneous cosmological model, although space and time aren't really independent or uniquely specified.

1.1.1 Absolute versus Relative Space

Newton claimed that since objects have absolute acceleration (whether something is accelerating is evidenced by non-inertial effects), they must have absolute velocity and position (even if only relative velocity and position are actually observable), so he believed in the existence of an absolute space that objects move with respect to. Others such as Leibniz, Berkeley, and later, Mach, claimed that acceleration, velocity, and position are always merely relative to other objects rather than with respect to any sort of pre-existing space, seeing space merely as the set of relations among objects rather than as a concrete substance that acts as a foundation for objects to exist within. While Newton claimed a single rotating object in an otherwise empty universe would show the non-inertial effects of its rotation, Leibniz claimed it made no sense to speak of a single object in rotation, and since no one could ever conduct the experiment, the argument has never been settled.

Einstein clearly set out with the notion of space as relative, which is evidenced by the fact that his theory ultimately became known as “Relativity.” Einstein’s notion of letting mass-energy define the spacetime structure in General Relativity seems to go in the direction of making the inertial properties of matter dependent on the contents of spacetime instead of existing independently like one would expect if space were absolute. Since a point mass falling in a gravitational field experiences no non-inertial effects, it is more natural to consider the particle to be unaccelerated and have no forces acting on it; thus, Einstein geometrized gravity to make the effect of falling in a gravitational field as natural as being in an inertial reference frame so that now gravity ceases to be considered a force and objects simply move inertially along geodesics in curved spacetime.

Interestingly, setting a shell of matter in rotation about a central mass in General Relativity will cause the central mass to itself experience non-inertial effects of rotation (Lense & Thirring 1918); however, the non-inertial effects will be smaller than if the central mass were rotated at the same angular velocity relative to the non-rotating shell of matter, so it isn’t simply a matter of relative rotation, although it does seem Machian at first. Also, a solution to Einstein’s Field Equations exists for an isolated black hole in rotation that is distinguishable from that of an isolated non-rotating black hole, which clearly contradicts the Machian notion that rotation must be relative to other objects, suggesting it is possible to have a single object in rotation in an otherwise empty universe. In fact, in General Relativity it is still possible to have vacuum spacetimes (such as Minkowski spacetime), which suggests space isn't merely a set of relations among objects, but must be something more foundational if it is still possible to have relations without objects.

Fundamentally, it appears it would be difficult to argue that rotation is merely relative to other objects rather than with respect to absolute space, since if it were equivalent to consider a rotating object to be a static object in a rotating universe, then at some finite distance from the object, the universe would appear to be rotating faster than the speed of light, in violation of the speed limit of Special Relativity: if we took the Earth to be static and watched the universe rotate by each day, we wouldn't have to look any further than the outskirts of the Solar System to see objects violating the speed of light.

Grünbaum (1964, 1974), Sklar (1974), and Feynman (1995) have claimed that the existence of absolute rotation in General Relativity stems from assuming boundary conditions at infinity. However, this appears to be a confusion. Boundary conditions don't need to be assumed at infinity to devise a solution to Einstein's Field Equations: the mass-energy source defines the entire spacetime structure, even if the source is bounded and the spacetime is infinite in extent. From Sklar's references, it appears this notion that absolute rotation depends upon boundary conditions at infinity originates in a hypothesis of Wheeler's (1964).

Wheeler hypothesized that it isn't sufficient that Einstein's Field Equations be satisfied by a spacetime, but that certain boundary conditions should also have to hold for a spacetime to be accepted as a valid solution. Wheeler took it for granted that inertial properties are solely due to mass-energy (possibly not a valid assumption if we can still state what is or isn't inertial in a vacuum spacetime like Minkowski). He objected to non-closed spacetimes on the grounds that a localized source would have to be controlling the inertial properties at infinity yet that the spacetimes can only asymptotically become Minkowski at infinity (he appears to have confused Minkowski spacetime with what is inertial, despite that the point of General Relativity is to turn the non-inertial motions due to gravitational forces into inertial motions in a curved spacetime). Wheeler rejected the Schwarzschild metric as a physically-reasonable solution on these grounds.

Regardless of whether Wheeler's hypothesis is valid, his argument wasn't that boundary conditions for a given spacetime can be independently varied to allow the inertial properties and the existence of absolute rotation to be arbitrarily imposed. He simply hypothesized that boundary conditions should be imposed on what is to be considered a physically-reasonable spacetime in determining whether to accept a given spacetime as a valid solution or not. Judging by Feynman's discussion (1995) the confusion appears to have arisen due to the fact that the Lense-Thirring effect (1918) was calculated using the weak-field limit, so asymptotic flatness was assumed in that case for solving the problem of objects in rotation. However, the weak-field limit is an approximation: in coming up with exact solutions of Einstein's Field Equations, asymptotic flatness need not be assumed in deriving asymptotically-flat spacetimes like Schwarzschild or Kerr (e.g. see the derivations in D'Inverno 1992) although asymptotic flatness is often assumed simply to obtain the solutions more quickly.

1.1.2 Asymmetric Time versus Asymmetric in Time

While it is generally assumed that our universe is spatially homogeneous (on large enough scales at least), it is also generally assumed that our universe isn't only time-inhomogeneous, but globally time-asymmetric. Before the expansion of the universe was discovered, Boltzmann (e.g. see Reichenbach 1956) had hypothesized that since entropy only increases (or stays the same) on average, then if the universe were infinite in time, an extremely low-entropy fluctuation would inevitably occur at some time, and people living in one side of the entropy fluctuation would observe time to be asymmetric with entropy increasing in one direction even though the universe wasn't time-asymmetric as a whole.

After the expansion of the universe was discovered, Gold (1962) assumed the expansion must itself bring about increasing entropy, since the redshift due to the expansion of the universe would prevent stars from ever being in thermal equilibrium with the radiation. However, as Davies (1974) has pointed out, if the expansion of the universe were to suddenly reverse itself, the contraction of the universe would lead to a blueshift, but it wouldn't immediately impact the thermodynamic processes in the stars; thus, the expansion of the universe can't be directly tied to the entropy arrow of time. It is that the universe began in a low entropy state, not that the expansion of the universe is associated with increasing entropy, that is relevant. Thus, the expansion and entropy arrows of time appear to be separate and an explanation such as Boltzmann's isn't sufficient to explain both. This leaves a problem as to whether time is inherently time asymmetric or events are merely unfolding asymmetrically in time due to some constraint on the initial conditions.

Gravity tends to clump matter, so qualitatively it makes sense that gravitational entropy is minimized when matter is homogeneously distributed (although no quantitative description of gravitational entropy yet exists), which means the relatively homogeneous initial distribution of matter in the universe would have been of very low entropy. Penrose (1979) devised the Weyl Curvature Hypothesis stating that initial singularities must have no Weyl curvature (the relativistic equivalent of tidal forces, which would be associated with clumped matter distributions) and final singularities must have infinite Weyl curvature, so this hypothesis presupposes some type of time-asymmetry that would allow initial and final singularities to be distinguished.

The best evidence for an intrinsic asymmetry in time is the asymmetry in decay rates between the neutral kaon and anti-kaon, but this presupposes that an anti-particle is the same as a particle moving backward in time. In reality nothing can literally move through time (although our perception may suggest otherwise). If anything were to move through time, $1\ \mathrm{s\ s^{-1}}$ is dimensionless as far as being a rate of motion; and if the contents of time were always moving into the future and leaving the past vacant, then this would require events in time to change as they went from being future to present to past events. To say that a particle moves forward in time and *then* moves backward in time requires having a second timeline to be able to say what happens first. The direction of motion could never be distinguished within one timeline, so it should not be possible to distinguish a particle from an anti-particle if anti-particles really were particles moving backward in time. It would be more fair to say objects have extension in time than to say they are moving forward or backward in time. Thus, the decay asymmetry in the neutral kaons might just demonstrate an asymmetry between matter and anti-matter rather than a temporal asymmetry.

If time isn't itself asymmetric, and its contents are merely arranged asymmetrically in time, it is difficult to explain the low gravitational entropy of the early universe. It may be that since the entropy in the radiation was maximized by being homogeneous (radiation is governed by pressure and tends to spread itself out uniformly), and since radiation dominated prior to matter, then the gravitational entropy was forced to be low. Another alternative is that it may not be possible to have a Big Bang singularity unless the universe began homogeneous, so if the universe formed from a single event, it might have to be initially homogeneous. Yet another alternative is that a large ensemble of possible universes exists, in which case some universes would randomly happen to have low entropy, allowing for the existence of a thermodynamic arrow and potentially the evolution of life, so then it would not be unexpected that we would happen to find ourselves in a universe that began with low entropy (this argument is an example of the anthropic principle, which requires that the universe must have properties that are consistent with the existence of human beings). It remains a mystery as to why exactly the universe began in such a low entropy state if it isn't intrinsically time-asymmetric.

Finally, it should be noted that since the spacetime and the mass-energy aren't independent in General Relativity, in the same way a point mass modifies the curvature of what would normally be flat Minkowski spacetime and introduces a radial asymmetry, the contents of spacetime could introduce an asymmetry in time, so it might not make sense to ask whether time itself is asymmetric or only its contents are. Clearly if time itself is asymmetric, its contents can be forced to be so, but it is also possible that time not be intrinsically asymmetric yet for an asymmetry in its contents to modify it into being asymmetric.

1.2 The Influence of Inhomogeneities on the Universe's Expansion

1.2.1 Theoretical Motivations

Raychaudhuri's equation is

$$\theta_{|a}u^a = -R_{ab}u^a u^b - \frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} + a^a{}_{||a}, \tag{1.1}$$

which essentially states that the partial derivative of the expansion θ goes as the negative of an energy density term, minus an expansion term, minus a shear term, plus a vorticity term, plus an acceleration term. Thus, mass-energy tends to decrease the universe's expansion (as we are commonly aware of in considering the critical density for the universe), shear also tends to decrease the expansion (the volume expansion is maximized when the universe expands isotropically and will be diminished if there is shear), and vorticity tends to increase the expansion (which we are familiar with in the case of rotating objects like spiral galaxies that maintain themselves against gravitational collapse). If the fluid flow is geodesic, there will be no acceleration; otherwise, acceleration (or deceleration) will not surprisingly increase (or decrease) the expansion.

The components in Raychaudhuri's equation can be understood by taking the covariant derivative of the velocity field u^a and decomposing it (e.g. see Stephani 1990) as

$$u_{a||b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\theta h_{ab} - a_a u_b, \tag{1.2}$$

where ω_{ab} is the rotation tensor, σ_{ab} is the shear tensor, θ is the expansion, h_{ab} is the projection tensor, and $a_a = u_{a||c}u^c$ is the acceleration. The rotation tensor is the antisymmetric part of $u_{a||b}$,

$$\omega_{ab} = u_{[a||b]} + a_{[a}u_{b]}, \tag{1.3}$$

which represents vorticity. The expansion tensor is the symmetric part of $u_{a||b}$,

$$\theta_{ab} = u_{(a||b)} + a_{(a}u_{b)}, \tag{1.4}$$

and (since $a^a u_a = 0$) is related to the expansion term

$$\theta = u^a{}_{||a}, \tag{1.5}$$

which is the term most closely related to the Hubble constant, representing the direction-independent expansion that an observer would see. The projection tensor is

$$h_{ab} = g_{ab} + u_a u_b, \tag{1.6}$$

where g_{ab} is the metric tensor. The shear tensor is the symmetric and trace-free part of $u_{a||b}$,

$$\sigma_{ab} = \theta_{ab} - \frac{1}{3}\theta h_{ab}, \tag{1.7}$$

which represents the direction-dependent gradient of the velocity field with the mean expansion taken out (such that it just shows anisotropy, with the universe having increased or decreased expansion in certain directions).

Assuming an inhomogeneous universe contains the same overall mass-energy as a homogeneous universe, just redistributed, and that the fluid flow is either geodesic or regions of accelerated or decelerated acceleration tend to cancel out, then Raychaudhuri's equation suggests the presence of inhomogeneities in the universe may affect its overall evolution from that of a perfectly homogeneous universe by introducing shear in the velocity field. With matter tending to fall toward an overdensity such that the tidal influence on the volume expansion of particles will be to increase their expansion in the direction toward the overdensity and decrease their expansion in the directions perpendicular to that of the overdensity, the volume expansion will be decreased. With a universe filled with many overdensities and underdensities, the net effect would be many sheared volumes with decreased volume expansion, meaning the overall volume expansion of the universe should be decreased.

Raychaudhuri's equation also suggests the presence of vorticity could actually increase the universe's volume expansion, tending to act like a cosmological constant. However, since the presence of vorticity only seems to be apparent in systems on the scales of solar systems and spiral galaxies, it appears vorticity is only relevant regionally for supporting specific objects against collapse, rather than on a global scale. While it seems unlikely the universe could have any net rotation, as discussed by Gödel (1949) a non-expanding universe of density 10^{-30} g cm^{-3} would only have to undergo rotation every 2×10^{11} yrs to be completely supported by rotation, so if it were possible for even the most negligible amount of rotation to be introduced, it could be quite significant.

Interestingly, Raychaudhuri (1955) showed that if vorticity vanishes and the local expansion is isotropic, then space is locally isotropic. However, we know that in reality space can't be locally isotropic since inhomogeneities curve space so that planets orbit the Sun, we observe gravitational lensing, etc. Thus, assuming that vorticity can be neglected in considering the influence of inhomogeneities, then that space is locally anisotropic shows its expansion must be locally anisotropic, so it isn't strictly correct to interpret the universe's expansion as being a uniformly expanding space with the galaxies having peculiar velocities with respect to this uniform expansion.

Since the spacetime structure is tied to the mass-energy distribution in General Relativity, it probably isn't that surprising that the universe's expansion would be considered to be locally inhomogeneous if the matter is locally inhomogeneous. However, it is more than just a question of whether to interpret the universe's expansion as spatially uniform with matter having peculiar velocities in space versus saying the expansion is non-uniform. It isn't in general equivalent that the spacetime corresponding to a homogeneous mass distribution can be used in place of the spacetime corresponding to an inhomogeneous mass distribution, even if the spatial average of the inhomogeneous mass distribution is the same as the homogeneous spacetime (e.g. see chapter 9 of Krasiński 1997). This is because Einstein's Field Equations don't equate the spacetime metric g_{ab} directly to the mass-energy T_{ab}, but instead they are related by

$$R_{ab} - \frac{1}{2} R g_{ab} = -\kappa T_{ab} \tag{1.8}$$

(where the Ricci tensor R_{ab} and scalar R both come from taking derivatives of g_{ab}). Thus, to use spatially-averaged quantities and have them satisfy the Field Equations, one would need the spatial average of the left-hand side, which would involve knowing the exact form of g_{ab} and calculating R_{ab} and R before performing the spatial average, so just using the homogeneous metric g_{ab} that corresponds to a homogeneous T_{ab} to represent the spatial average of an inhomogeneous T_{ab} will not generally be consistent. If the homogeneous metric is used to calculate the left-hand side and the left-hand side is then equated to the spatially-averaged inhomogeneous T_{ab}, in reality there will generally be a difference between the two sides of the equation, which can be interpreted as having the effect of a cosmological constant (this will be discussed further in Chapter 2).

While many people prefer to interpret the universe's expansion as homogeneous and then add on the peculiar velocities of objects, they do at least grant that gravitationally-collapsed objects like stars and galaxies don't participate in the expansion, as these objects have reached turnaround, gravitationally collapsed, and then virialized, so the matter they are composed of no longer possesses the velocity field associated with the expansion of the universe. Strangely, some have used the interpretation of a uniformly expanding space and assumed that systems on all scales continue to expand along with the universe. Dumin (2003) used lunar radar to measure the rate that the Moon is spiralling away from the Earth, and then compared that with the rate expected due to the transfer of energy from the Earth's rotation to the Moon to find an excess, which he calculated to correspond to a value for the Hubble constant H_o of 33 km s^{-1} Mpc^{-1}.

Previously, Gautreau (1984) showed that orbits about a mass in an FRW universe wouldn't be circular although they would deviate only slightly; however, this is to be expected even in Newtonian gravity, as the expansion of the universe would cause matter in the background universe to stream out of the orbit and require the orbit to spiral outward. Assuming in reality that the matter of the background universe went into the collapsed objects or has at least been perturbed so greatly from the velocity field of the background universe in the vicinity of the collapsed objects that it wouldn't actually stream outward, then one wouldn't expect any flux of matter out of the orbits and presumably they would then be stable.

However, in calculating the size of a hydrogen atom in an expanding universe, Bonnor (1999) found that it would expand 10^{-67} as fast as matter comoving with the expansion of the universe, which however slight, is still almost seven times as much as the expected value based on a Newtonian calculation of the streaming of the background matter of the universe out of the atom. While it might not actually make sense to talk about the matter of the universe streaming out of an atom, in General Relativity a homogeneous matter distribution is smooth down to all scales rather than consisting of individual particles, so in theory the calculation makes sense. Bonnor suggested that this excessive expansion might be due to something analogous to frame dragging, so in the same way a rotating object induces slight vorticity in surrounding matter, the expansion of the universe might induce slight expansion in non-expanding objects. It seems possible a factor of 2π could have simply been dropped somewhere in performing the Newtonian calculation. However, if the relative motion of matter does influence its inertial properties, then the possibility of induced expansion seems plausible. It should be noted that if the induced expansion is of order 10^{-67} of the expansion of the universe, clearly the effect would be so negligible as to be unobservable, so it wouldn't make measurements of H_o on collapsed systems reasonable, leaving measurements such as Dumin's (2003) as being highly doubtful.

1.2.2 Observational Evidence

There are several areas of prior research that are pertinent to inhomogeneities and variation in the universe's expansion. In this section voids, CMB anisotropy, bulk flows, and local anisotropy in H_o will be discussed.

Existence of Voids

Some people (e.g. Lerner 1991) have argued against the Big Bang theory on the grounds that it would have taken 100 billion years for the galaxies to fall out of the voids, assuming they fell at velocities on the order of that of the usual peculiar velocities of the galaxies. However, considering the universe to be inhomogeneous, it is much easier to explain the relative absence of galaxies in the voids if they simply never formed there because the voids are underdense regions that expand fast enough to never have large regions undergo collapse. The voids aren't vacuous, so it doesn't make sense to expect that galaxies would have fallen out of them yet that they would still have a background density due to diffuse matter that should have just as easily fallen out if the galaxies had.

If the universe initially varied slightly from a density parameter of $\Omega = 1$, it would either have expanded so fast that galaxies never formed, or it would have collapsed a long time ago. The very existence of the universe's large network of voids and superclusters seems best explained if the voids were the initially underdense regions of the universe that acted like an $\Omega < 1$ universe, so that the density slowed the expansion very little and allowed the voids to balloon up without structure formation occurring within them; and if the superclusters were the initially overdense regions of the universe that acted like an $\Omega > 1$ universe, allowing them to reach turnaround, and in the densest areas collapse to create a hierarchy of structure.

If one wanted to interpret the expansion of space as uniform, then the existence of large-scale structure would require space to be undergoing a flux out of the collapsed regions, where the objects are no longer expanding, and into the voids. Neighbourhoods of structure would have to be considered to be making bulk motions with respect to the uniformly expanding space, so the notion of peculiar velocities on a uniformly expanding space really doesn't work as well as if all the galaxies were uniformly distributed through the universe and had totally random peculiar velocities relative to one another. Thus, the notion of a uniform Hubble flow with peculiar velocities doesn't appear to be a particularly good interpretation given the hierarchy of structure that exists.

Moffat & Tatarski (1995) looked at what observational effects we would theoretically observe if we were to inhabit a local void. Via comparison of their theoretical curves with a survey of redshift-distance determinations, they found the data were better fit by a model with a local void than by a homogeneous universe. Zehavi et al. (1998) used 44 Type Ia supernova H_o values to show that we may just inhabit an underdense region of the universe (where the expansion in the velocity field has been slowed less due to gravity than in more dense regions of the universe). Referring to fig. 4 of Freedman et al. (2001), it appears that the H_o values tend to fall off beyond a distance of 100 Mpc, which suggests the universe may be expanding faster locally. A here-there difference in the universe's expansion could be an alternative to the notion of a now-then difference, which is the assumption the accelerating universe (Perlmutter et al. 1999) rests on.

CMB Anisotropy

Tegmark, Oliveira-Costa, & Hamilton (2003) found a correlation in alignments of the CMB quadrupole and octupole. Suspiciously, the maxima and minima tend to lie in a plane and the poles of the planar quadrupole and octupole also align with the CMB dipole (although Tegmark et al. appear to try to disguise this additional correlation by reporting the co-ordinates with a negative longitude). It seems likely a common influence is at work in creating the dipole/quadrupole/octupole correlation, suggesting the CMB dipole might have more to do with the influence of large-scale structure on the velocity field, rather than simply being the result of our own locally-perturbed peculiar motion. That this influence brings about a noticeable component of the quadrupole/octupole suggests it extends from a large scale.

As an example, if a large overdensity existed that we were falling toward, that overdensity should also be tidally shearing the velocity field in our neighbourhood as objects in our neighbourhood fell toward the overdensity. For the shearing of the velocity field to be significant enough to show up in the CMB, one would expect the neighbourhood undergoing this shear would have to be reasonably large, suggesting the inhomogeneity at work would have to be reasonably distant. Thus, it could suggest our peculiar motion with respect to the CMB is due to inhomogeneity on a large-scale, rather than merely being a local perturbation in the velocity field.

However, the planar quadrupole/octupole pattern observed is obviously not consistent with the type of shear that would be generated by an essentially spherical overdensity, as that type of shear wouldn't introduce highs and lows in a plane perpendicular to the direction we were falling along. Planar variations could be introduced if we were falling toward a non-spherical overdensity such as a filament (or the edge of a wall), with the velocity field getting compressed perpendicular to the length of the overdensity and being unaffected parallel to the length of overdensity. With some type of complicated structure such as a supercluster with intersecting filaments/walls, then a planar quadrupole/octupole pattern might be possible, especially if the object were large and extended such that the mass within a given angle increased with distance to compensate for differences in distance so that relatively little shear existed along the CMB dipole direction.

Bulk Flows

Bulk flow studies involve determining the peculiar velocities (with respect to the CMB) of galaxies within a sample volume to determine a net streaming motion for that volume (or of a sample volume with respect to us; if that velocity isn't the opposite of our motion with respect to the CMB, then it is equivalently saying that the sample volume is moving with respect to the CMB). Lauer & Postman (1994) determined a velocity for the Local Group with respect to an Abell cluster sample extending out to a recessional velocity of 15,000 km s^{-1}. Rather surprisingly, this velocity differed from the velocity of the Local Group with respect to the CMB, suggesting a net velocity of the Abell cluster sample (with respect to the CMB) of 689 ± 178 km s^{-1} toward $l = 343^{o}$, $b = +52^{o}$. Although other bulk motion studies haven't yielded exactly the same direction as the Lauer & Postman result, other bulk motion studies have obtained directions that correlate with each other more than they do with the Lauer & Postman study (see Table 1.2.2). Zaroubi (2002) provided a thorough review of bulk flows, showing that there is agreement for sample volumes

Table 1.2.2: Bulk Flow Data

cz_{max} (km s^{-1})	Velocity (km s^{-1})	Co-ordinates (l, b)	Reference
6,000	220±60±50	(304±16, 25±11)	da Costa et al. 2000
9,200	310±120	(337, -15) ± 23	Giovanelli et al. 1998
10,000	336±96	(321, -1)	Parnovsky et al. 2001
11,000	370±110	(305, 14)	Dekel et al. 1999
12,000	687±203	(260±13, 0±11)	Hudson et al. 2004
13,000	700±250	(272, 10) ± 35	Willick 1999
15,000	689±178	(343, 52)	Lauer & Postman 1994

less than 60 h^{-1} Mpc in radius, but beyond that only half the studies find a bulk flow with a magnitude consistent with the expected falloff, while the other half find a bulk flow of roughly three times the expected magnitude.

Over a large enough sample volume, the relative mass fluctuations should be small enough for there to be little peculiar motion of the volume with respect to the CMB. Assuming the universe approaches homogeneity on larger scales, the existence of a 700 km s^{-1} bulk flow on scales of ∼120 h^{-1} Mpc is very surprising, as that is as high as a bulk flow could be expected to be even in a very localized volume. Colless et al. (2001) have argued that the higher-than-expected bulk flows specifically result from the error due to the window functions of the samples: correcting for the window function of their own sample, they obtain an insignificant bulk flow of 159 ± 158 km s^{-1} (no direction reported). Still, the fact that there appears to be good agreement among the directions reported from the bulk flow studies that do find significant flows and even with the directions of the studies that don't (see fig. 1 of Zaroubi 2002) suggests that these net streaming motions may not just be artifacts of observational biasing as Colless et al. have suggested. Hudson et al. (2004) have also claimed the high-magnitude bulk flows result from sparse sampling, yet apparently found no way to take the error due to this into account in reporting their bulk flow result.

It is difficult to say that either the expected-magnitude or high-magnitude bulk flow studies are tainted when they all manage to yield essentially the same direction, so whether the bulk flows are falling off with sample volume as expected remains uncertain. Regardless, even the expected bulk flow of ∼200 km s^{-1} on scales of 100

h^{-1} Mpc is likely larger than most people would intuitively expect, suggesting the notion of peculiar motions on a uniformly expanding background universe isn't the most useful model of the universe's expansion. Also, the bulk flow studies are overly simplistic to consider a sample volume to be moving coherently in one direction when there should be multiple perturbations to the velocity field from that of a homogeneous expansion: the bulk flow studies overlook all the individual variations that bring about the net streaming motions of consideration (or that could cancel each other out across the sky to yield no bulk flow of the sample volume).

Local Anisotropy in H_o

The most accurate work to date to study H_o is the *HST* Extragalactic Distance Scale Key Project, which finally (Ferrarese et al. 2000; Gibson et al. 2000; Kelson et al. 2000; Mould et al. 2000; Sakai et al. 2000; Freedman et al. 2001) yielded distances accurate enough for a meaningful study of real variation in observed values of H_o. McClure and Dyer (2004) used these H_o values and the directions they were obtained along to map out how H_o varies with direction on the sky and found this variation to be statistically significant. This variation was at least in partial agreement with the bulk flow directions (see Figure 1.1), with bulk flow directions tending to align with higher H_o regions of the sky; however, the pattern of H_o variation observed across the sky wasn't consistent with a simple dipole or bulk flow motion, suggesting the bulk flow studies really may be overly simplistic in failing to discern more complicated effects than a simple bulk flow in the velocity field.

While this mapping method makes sense for demonstrating there are directional variations in the universe's local expansion, it looks at recessional velocity per unit distance for objects that are different distances away, which is a somewhat clouded way of looking at the velocity field. Some work was done to examine how the directional variation may also vary with distance by mapping the variation on the sky using data binned at different distances. These maps show a statistically significant directional variation that decreases in magnitude with distance, as well as another directional variation (of questionable statistical significance) that remains constant in magnitude with distance. (Neither of these variations appears to change direction with distance.)

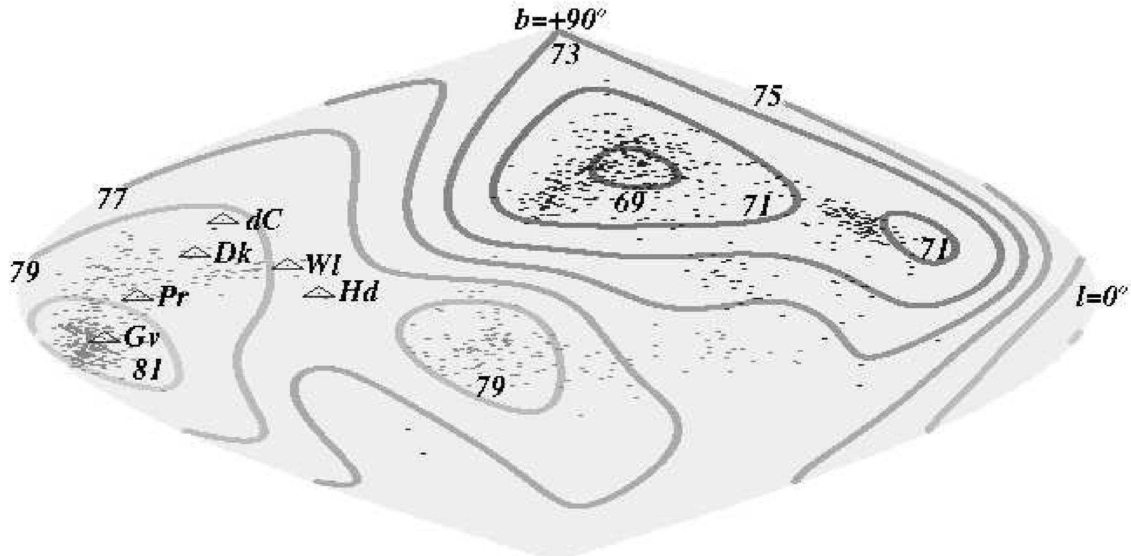

Figure 1.1: Hubble constant contour map (in Galactic co-ordinates) for a smearing of the Key Distance Project H_o values across the sky, shown with 500 randomized map extrema (dots) and bulk flow determinations (triangles). Contours for the smeared-out map of the actual H_o data range from low (dark) to high (light) values of H_o (in km s^{-1} Mpc^{-1}) as labelled, and positions of the randomized minima and maxima, which are calculated using Gaussian deviates to randomly tweak all the H_o values about their ranges of uncertainty, are indicated by dark and light dots respectively. From high to low latitude, the bulk flows are those of da Costa et al. (2000), Dekel et al. (1999), Willick (1999), Hudson et al. (2004), Parnovsky et al. (2001), and Giovanelli et al. (1998).

Chapter 2

Preliminaries

2.1 Solution Generating Techniques

A solution to Einstein's Field Equations

$$R_{ab} - \frac{1}{2} R g_{ab} = -\kappa T_{ab} \tag{2.1}$$

can either be obtained by finding a metric g_{ab} from which the entire left-hand side can then be calculated to allow the energy-momentum tensor T_{ab} to be determined, or by starting with an energy-momentum tensor and trying to work backward to obtain the metric that corresponds to it. However, the choice of any particular metric may not lead to a physical energy-momentum tensor, as the energy-momentum tensor may not satisfy energy conditions or be describable in terms of any known energy-momentum components; and while it would be usual to want to determine the spacetime corresponding to a particular mass-energy distribution, starting with the energy-momentum tensor doesn't easily allow the metric to be calculated, especially in cases where specifying the mass-energy distribution requires the metric to be known in order to be able to specify the mass-energy distribution in the first place.

In coming up with new solutions, it is most straightforward to transform known metrics in ways that will ideally lead to physically-interesting energy-momentum tensors. Common transformation methods are conformal transformations and Kerr-Schild transformations, which will be discussed in Sections 2.1.1 and 2.1.2. It is also possible to cut and paste known spacetimes together to obtain more complex mass-energy distributions without having to determine new metrics, which will be discussed in Section 2.1.3; however, there are strict conditions on the matchings.

To be considered a solution of the Field Equations, a metric must yield an energy-momentum tensor that corresponds to a physically-possible source. The conditions that are usually considered are the weak, strong, and dominant energy conditions (e.g. see Wald 1984). The weak energy condition requires that the energy density not be negative. An observer with unit timelike 4-velocity u^a will measure the energy density as $T_{ab}u^a u^b$, so $T_{ab}u^a u^b \geq 0$ for the weak energy condition and the conditions for the energy density μ and the pressures along the principal directions p_i are that $\mu \geq 0$ and $\mu + p_i \geq 0$. From Einstein's Equations it can be shown that

$$R_{ab}u^a u^b = -\kappa \left(T_{ab}u^a u^b + \frac{1}{2}T \right). \tag{2.2}$$

The strong energy condition requires $T_{ab}u^a u^b \geq -\frac{1}{2}T$ such that the stresses of matter are not too large and negative (which requires that $\mu + p_i \geq 0$ and $\mu + \Sigma p_i \geq 0$). The dominant energy condition requires that $T^a_b u^b$ be a timelike or null vector (which requires $\mu \geq |p_i|$) so that the observer sees matter flowing no faster than the speed of light. If a spacetime violates the energy conditions in one region, it doesn't invalidate the spacetime as a whole: the fact that it is possible to cut and paste different spacetimes together suggests it may be possible to cut out the invalid regions of spacetime and replace them with physically-acceptable regions, so the regions of spacetime that are valid are still useful on their own.

2.1.1 Conformal Transformations

Conformal transformations

$$\bar{g}_{ab} = \Omega^2 g_{ab} \tag{2.3}$$

can be used to generate a new metric $\bar{g}_{ab}$ by taking a known metric g_{ab} and performing a point-dependent rescaling of the original metric via the conformal factor Ω. Interestingly, taking the Robertson-Walker metric for the Einstein-de Sitter universe (the FRW universe that has flat spatial sections)

$$\mathrm{d}s^2 = -\mathrm{d}t_c^2 + [R(t_c)]^2 \left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2) \right) \tag{2.4}$$

and changing from the cosmological time t_c to a new time co-ordinate t via $R(t)\mathrm{d}t = \mathrm{d}t_c$, then

$$\mathrm{d}s^2 = [R(t)]^2 \left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\right), \tag{2.5}$$

which is just a conformal transformation of the Minkowski metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2). \tag{2.6}$$

Conformal transformations can also transform Minkowski spacetime into non-flat FRW spacetimes, but the conformal factor is slightly more complicated.

Conformal transformations preserve conformal curvature in the form of the Weyl tensor $C^a{}_{bcd}$ (the relativistic equivalent of tidal forces), which corresponds to the trace-free part of the Riemann curvature tensor $R^a{}_{bcd}$. Conformal transformations also preserve null geodesics so that the causal structure of the original and transformed spacetimes agree. The Ricci scalar and tensor aren't conformally invariant however, so it is possible to introduce mass-energy, as seen in the example above in going from a vacuum spacetime to a radiation-filled or dust-filled Einstein-de Sitter universe. While it isn't possible to introduce shear or rotation in the velocity field by way of a conformal transformation, it is possible to introduce expansion, as seen in the above example going from Minkowski spacetime to the Einstein-de Sitter universe.

If one is interested in obtaining cosmological models with Weyl curvature or shear, conformal transformations would only be useful for taking models that already have Weyl curvature or shear and transforming them to introduce expansion to obtain models that exist as part of cosmological models. Since Minkowski spacetime is conformally related to the Einstein-de Sitter universe, this suggests that spacetimes such as Schwarzschild that are asymptotically Minkowski can be transformed with the same conformal factor to obtain cosmological counterparts that are asymptotically Einstein-de Sitter. Previously, Thakurta (1981) and Sultana & Dyer (2005) used this approach to yield cosmological Kerr and Schwarzschild black holes (although solutions for the source only exist in the Schwarzschild case).

2.1.2 Kerr-Schild Transformations

Kerr-Schild transformations (Kerr & Schild 1965; see also Stephani et al. 2003)

$$\bar{g}_{ab} = g_{ab} + 2Hl_a l_b \tag{2.7}$$

can be used to generate new metrics by taking a known metric and adding a component based on a scalar field H and null geodesic vector field l_a. The scalar field is undetermined and is only subject to the constraint that it result in a physical energy-momentum tensor. The null geodesic field used in the transformation will remain a null geodesic field of the transformed metric since

$$\bar{g}_{ab} l^a = g_{ab} l^a, \tag{2.8}$$

but this relation won't generally be obeyed by other null vectors, so the causal structure isn't totally preserved. The energy-momentum tensor transforms according to

$$l^a \bar{T}_{ab} = l^a T_{ab} + F l_b, \tag{2.9}$$

where F is a scalar field, so the transformation adds a component that can be localized in spacetime via the scope of the scalar field.

Most notably, this transformation can be used to obtain the Kerr metric (for a rotating black hole) from the Minkowski metric with

$$2H = \frac{2mr^3}{r^4 + a^2 z^2} \tag{2.10}$$

and

$$l_a = \left(1, \frac{rx + ay}{a^2 + r^2}, \frac{ry - ax}{a^2 + r^2}, \frac{z}{r}\right), \tag{2.11}$$

which means the Schwarzschild metric ($a = 0$) is also a Kerr-Schild transformation of Minkowski space with

$$2H = \frac{2m}{r} \tag{2.12}$$

and

$$l_a = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right), \tag{2.13}$$

or in spherical coordinates with $l_a = (1, 1, 0, 0)$. Thus, the Schwarzschild metric can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2) + \frac{2m}{r}(\mathrm{d}t + \mathrm{d}r)^2. \tag{2.14}$$

This is equivalent to the Eddington-Finkelstein form of the Schwarzschild metric, which is obtained by taking the standard form of the Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}\bar{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\;\mathrm{d}\phi^2) \tag{2.15}$$

and performing the transformation

$$\mathrm{d}\bar{t} = \mathrm{d}t - \frac{2m}{r-2m}\mathrm{d}r. \tag{2.16}$$

Analogous to the Kerr-Schild transformation used to obtain the Schwarzschild metric, the Reissner-Nordström metric (for a charged black hole) can be obtained with

$$2H = \frac{2m}{r} - \frac{e^2}{r^2} \tag{2.17}$$

and the same null vector field l_a to yield

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\;\mathrm{d}\phi^2) + \left(\frac{2m}{r} - \frac{e^2}{r^2}\right)(\mathrm{d}t + \mathrm{d}r)^2. \tag{2.18}$$

If the null vector field is changed to $(1, -1, 0, 0)$ or $(-1, 1, 0, 0)$, the $\mathrm{d}t\mathrm{d}r$ cross term will change sign (all the other terms stay the same), so it is equivalent to doing a time inversion, which will yield white hole spacetimes instead of black hole spacetimes.

It is possible to introduce vorticity, shear, and Weyl curvature with a Kerr-Schild transformation, so Kerr-Schild transformations are useful for adding inhomogeneities into homogeneous spacetimes. If one is interested in obtaining cosmological models with Weyl curvature or shear, then Kerr-Schild transformations could be performed on FRW spacetimes to introduce inhomogeneities. Previously, Vaidya (1977) and Patel & Trivedi (1982) used this approach to obtain cosmological Kerr and Kerr-Newman black holes (although solutions for the source only exist in the Schwarzschild or Reissner-Nordström limits).

2.1.3 Spacetime Matchings

Another way to generate new spacetimes is to cut and paste known spacetimes together to create spacetimes with more complex mass-energy distributions. A common example is the Swiss cheese universe (Einstein & Straus 1945), which is constructed by cutting out spheres from an FRW universe and collapsing the matter down into Schwarzschild stars or black holes. Not just any spacetimes can be joined together: there are junction conditions that need to be satisfied for the spacetimes to be matched. The junction conditions commonly used to match spacetimes are those of O'Brien & Synge (1952), Lichnerowicz (1955), or Darmois (1927).

The Darmois conditions require that the first and second fundamental forms match across the junction, and are generally the most useful junction conditions because they can be used on spacetimes where different co-ordinates are used on opposite sides of the junction. The first fundamental form is

$$\Omega_{\alpha\beta} = g_{ab}\frac{\partial x^a}{\partial u^\alpha}\frac{\partial x^b}{\partial u^\beta}, \tag{2.19}$$

which is the 3-space metric inherited from the spacetime the matching surface, assumed non-null, is embedded in, and the second fundamental form is

$$\Upsilon_{\alpha\beta} = -n_{a||b}\frac{\partial x^a}{\partial u^\alpha}\frac{\partial x^b}{\partial u^\beta}, \tag{2.20}$$

which describes the derivative of the unit normal vector to the hypersurface. The u^α co-ordinates are the co-ordinates of the 3-space of the hypersurface. The normal is given by

$$n_a = \frac{f_{|a}}{|g^{bc}f_{|b}f_{|c}|^{1/2}}, \tag{2.21}$$

where f is a function of the co-ordinates such that it is zero on the junction.

The O'Brien & Synge and Lichnerowicz conditions require the co-ordinates to be the same on both sides of the junction. The Lichnerowicz conditions require merely that the metric g_{ab} and its derivatives $g_{ab|c}$ match across the junction. While satisfying the O'Brien & Synge conditions is sufficient for satisfying the Lichnerowicz or Darmois conditions, the O'Brien & Synge conditions require that g_{ab}, $g_{\alpha\beta|0}$, and T^0_a all match across the junction, which is unnecessarily restrictive since x^0 need not be a temporal co-ordinate, and in general we wouldn't expect that energy-momentum distributions need to be continuous in space in the same way that we expect continuity in time. Thus, the Lichnerowicz conditions are preferable to the O'Brien & Synge conditions and are useful when the co-ordinates are the same on both sides of the junction so that the Darmois conditions aren't needed.

2.2 Previous Work on Cosmological Black Holes

2.2.1 Swiss Cheese Black Holes

The most basic cosmological black holes are Swiss cheese (Einstein & Straus 1945) black holes that are constructed by matching a Schwarzschild exterior onto a surrounding dust-filled FRW universe. The Lemaître-Tolman-Bondi spacetimes (Lemaître 1933; Tolman 1934; Bondi 1947) describe more general spherically-symmetric spacetimes, and could be used to match a Schwarzschild black hole onto FRW via an underdense intermediate region that smoothly matches onto an FRW region, which seems more realistic than a pure vacuum region that abruptly matches onto FRW.

In some sense, the Swiss cheese black holes are too perfect because the overdense and underdense regions are balanced exactly such that the external FRW universe is completely uninfluenced by them, which makes them uninteresting if one is interested in the possible influence of inhomogeneities on the universe. Because the FRW region of the universe is totally uninfluenced by the black holes, it does make it possible to cut out many Swiss cheese holes and construct an exact model of a universe with multiple black holes though, which would practically be impossible to achieve in a spacetime where the influence of a black hole extends throughout the universe.

2.2.2 Kerr-Schild Cosmological Black Holes

Vaidya (1977) and Patel & Trivedi (1982) have obtained metrics corresponding to Kerr and Kerr-Newman black holes superimposed with FRW universes by performing Kerr-Schild transformations on closed FRW universes. Thakurta (1981) and Sultana & Dyer (2005) have obtained metrics corresponding to Kerr and Schwarzschild black holes superimposed with Einstein-de Sitter universes by performing conformal transformations of black hole spacetimes (Thakurta doesn't perform the transformation on the Kerr-Schild form of the black hole spacetime as Sultana & Dyer do, however).

Since a black hole is a Kerr-Schild transformation of Minkowski space, and FRW is a conformal transformation of Minkowski space, it is clear one can start with Minkowski space, do a Kerr-Schild transformation to get a black hole, and then perform a conformal transformation to get a black hole in an FRW background.

Alternatively, one can start with Minkowski space, do a conformal transformation to get an FRW universe, and then perform a Kerr-Schild transformation to get a black hole in an FRW background. Thus, both approaches are similar, the only difference being whether the Kerr-Schild part of the metric contains the conformal factor or not. The same solution can be obtained either way if the scalar field H contains the conformal factor when performing the Kerr-Schild transformation after the conformal transformation.

Comparing the Vaidya-type solutions with the Thakurta-type solutions, the Kerr-Schild part of the metric for the Vaidya-type black holes doesn't contain the conformal factor, so the Vaidya-type solutions differ in that the Kerr-Schild scalar field isn't time-dependent. Thus, in the Vaidya-type solutions, the component of the metric corresponding to the black hole doesn't participate in the expansion of the background universe.

In a practical sense, a Vaidya-type solution would be a good representation of a cosmological black hole that has collapsed out of an evolving mass overdensity in the universe such that the black hole no longer participates in the expansion of the universe (the drive toward finding Kerr solutions seems to in fact be motivated by the expectation that overdensities will realistically collapse down into rotating systems). However, the metric actually represents a black hole that has existed from the time of the Big Bang, so it really represents a black hole that was born not participating in the universe's expansion, which seems perhaps unlikely.

A Thakurta-type cosmological black hole seems more reasonable in that the universe simply starts with an inhomogeneity such that the black hole exists at time zero while it still participates in the expansion like the rest of the universe. However, these black holes will be less interesting to people who want solutions they can apply to black holes that have collapsed out of an initially homogeneous mass-energy distribution for the universe, and if Penrose's Weyl Curvature Hypothesis (1979) were true for some reason, then it wouldn't be realistic to expect that black holes could immediately exist from the moment of the Big Bang.

It should be noted that there is no interpretation of the energy-momentum tensor for the rotating cases of cosmological black holes. With the exception of Sultana & Dyer (2005), people have generally been content to speak on the basis of the metric looking like the superposition of a black hole with FRW, and have been somewhat remiss about carefully interpreting the energy-momentum tensor, which is important because a solution can't be considered a valid solution of Einstein's Field Equations unless the energy-momentum tensor has been interpreted as physically corresponding to something and the energy conditions are satisfied.

Interestingly, while it has been shown by Nayak, MacCallum, & Vishveshwara (2000) that it is possible to match the metric for a Schwarzschild black hole onto the surrounding cosmological Schwarzschild black hole spacetime of Vaidya at the stationary limit surface, Cox (2003) has shown that it isn't possible to match the metric for a Kerr black hole onto the surrounding cosmological Kerr black hole spacetime of Vaidya, suggesting Vaidya's cosmological Kerr metric isn't truly Kerr-like. Thakurta (1981) pointed out that the failure to obtain solutions for cosmological Kerr black holes isn't surprising considering there is no exact solution for a Kerr interior. Since a non-isolated Kerr black hole would tend to swirl the contents of the surrounding universe, then like a Kerr interior, it would require having a solution for an extended rotating source, rather than just a rotating singularity. The failure to obtain solutions for cosmological Kerr black holes and the failure to obtain a Kerr interior solution seem to be related: what exactly the difficulty is remains undetermined.

While solutions don't appear to exist for cosmological Kerr black holes, that still leaves the problem of obtaining cosmological Reissner-Nordström black hole solutions of the Thakurta-type. Also, while Sultana (2003) claimed to find a Schwarzschild solution for a black hole in a dust-filled Einstein-de Sitter universe, the metric considered was actually the white hole spacetime (the Kerr-Schild null-vector field was reversed), so the black hole case needs to be revisited. Solutions for radiation-dominated universes would also be of interest if one is interested in using these spacetimes as models of primordial black holes. Vaidya-type black holes in Einstein-de Sitter universes haven't yet been studied: the Vaidya-type black holes were all obtained in closed FRW universes, which although the radius of curvature can be set to infinity to obtain the metric for the flat case, the problem of interpreting the energy-momentum tensor in that case needs to be considered. Thus, Chapter 3 of this thesis will focus on interpretations of Kerr-Schild cosmological Schwarzschild and Reissner-Nordström black holes that are either comoving with the universe's expansion or not and exist in both radiation-dominated and matter-dominated Einstein-de Sitter universes.

2.2.3 Isotropic Cosmological Black Holes

The first cosmological black holes ever obtained were the Schwarzschild black holes of M^{c}Vittie (1933), which were eventually generalized to the Reissner-Nordström case by Gao & Zhang (2004). M^{c}Vittie's spacetime is similar to Thakurta's in that it is essentially a conformal transformation of the Schwarzschild metric, but written in isotropic form. The major difference from the Thakurta black holes is that the mass is a function of time such that it gets scaled down by the scale factor as the universe expands, so in that sense the M^{c}Vittie black holes look more like the Vaidya black holes, since the M^{c}Vittie black holes don't expand along with the universe either. However, the M^{c}Vittie spacetime is physically very different from that of Vaidya. Whereas Vaidya's solution yields an inhomogeneous energy density along with heat conduction, the M^{c}Vittie solution yields a homogeneous energy density and no heat conduction.

While M^{c}Vittie found that the pressure is isotropic, he neglected to consider whether it is physically reasonable or not, as Gao & Zhang (2004) also neglected to do in considering the charged case. Thus, new work analyzing where the M^{c}Vittie cosmological black holes satisfy the energy conditions will appear in Chapter 4 of this thesis. Also, new isotropic cosmological black hole spacetimes will be given for the case where the Schwarzschild mass isn't scaled down by the scale factor, and which yield different energy-momentum tensors than the Thakurta-type black holes.

2.3 Universes Containing both Radiation and Matter

To appropriately model the universe, a two-fluid model containing both radiation and matter is needed, as radiation governed the evolution for most of the first 100,000 years of the universe before it became matter dominated. In the radiation era the scale factor should have evolved with cosmological time like ${t_c}^{1/2}$, and in the matter era the scale factor should be evolving with cosmological time like ${t_c}^{2/3}$, assuming the universe can be approximated by an Einstein-de Sitter model.

Previously, two-fluid models that continuously vary between radiation domination and matter domination have been studied. This has been achieved either by specifying equations of state for the radiation and matter separately, and then solving an ordinary differential equation to obtain how the scale factor for the universe evolves with time, or by specifying a scale factor that changes its evolution with time and seeing what sort of variation between radiation domination and matter domination it physically corresponds to. As an example of the first approach, Jacobs (1967) obtained a simple model of a universe that smoothly evolves from radiation domination to matter domination by taking the energy density of the radiation to evolve with the scale factor R as R^{-4} while the energy density of the matter evolves as R^{-3}. This realistically models the way the universe would have evolved from radiation domination to matter domination, although it also never allows for any interchange of energy between radiation and matter whatsoever. As an example of the second approach, Coley (1985) suggested a scale factor that evolves with time as

$$R(t_c) = {t_c}^{1/2}(1 + h{t_c}^{1/(6b)})^b \tag{2.22}$$

so that at small times it evolves like ${t_c}^{1/2}$ and at later times like $h^b {t_c}^{2/3}$. Using this scale factor in the Robertson-Walker metric and then interpreting the energy-momentum tensor, depending on the value of b, Coley found that the rate of energy transfer between radiation and matter varies and can change sign, so different models of energy transfer between radiation and matter can be obtained by adjusting b.

In the case of cosmological black holes, the energy-momentum tensors become complicated, and it would be difficult to deal with a two-fluid background universe; yet at the same time, in order to devise solutions for primordial black holes, it necessitates having spacetimes that evolve from being radiation dominated to matter dominated. Ideally, it would be possible to match radiation-dominated spacetimes directly onto matter-dominated spacetimes across hypersurfaces of constant time, rather than having to bridge from one domain to the other and having a brief period of time when both components would simultaneously be significant. However, it is unknown whether it is possible to match regions of spacetime that suddenly change between radiation domination and matter domination, instantly changing from evolving as ${t_c}^{1/2}$ to ${t_c}^{2/3}$. Apparently no one has attempted to do this, most likely because homogeneous cosmologies are usually studied for which it is easily possible to use two-fluid models, and people would be more interested in modelling the gradual change from radiation domination to matter domination that the CMB suggests occurred in the real universe. Thus, the conditions for matching radiation domination to matter domination will be studied in Chapter 5 in the aim of generating solutions for primordial black holes that began in the radiation-dominated era.

2.4 The Backreaction of Inhomogeneities on the Universe's Expansion

As mentioned previously in Chapter 1, spatially averaging the energy-momentum tensor and using the metric that corresponds to the equivalent homogeneous energy-momentum tensor to represent the spatial average of the spacetime for the inhomogeneous mass-energy distribution isn't strictly correct, since looking at Einstein's Equations

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = -\kappa T_{ab} \tag{2.23}$$

it wouldn't be a spatially-averaged metric that corresponds to a spatially-averaged energy-momentum tensor. Expecting an inhomogeneous universe to evolve according to the spacetime for a homogeneous universe will generally yield a difference between the two sides of Einstein's Equations, which will appear to act like a cosmological constant and accelerate or decelerate the universe's expansion.

Several researchers have considered the problem of spatial averaging and the impact inhomogeneities may have on the evolution of the universe. Bildhauer (1990) used a pancake model, with structure formation occurring in only one dimension, and found that while the expansion is slowed in the direction of structure formation, the averaged scale factor actually grows faster. Further to Bildhauer (1990), Bildhauer & Futamase (1991) discussed the age problem and how an accelerating universe could account for the age discrepancy by making the true age of the universe older. Futamase (1996) used a different spatial averaging and also found that the universe is accelerated by inhomogeneities. Bene, Czinner, & Vasúth (2003) also attempted to use spatial averaging to show that the universe's acceleration may be caused by the backreaction of inhomogeneities on the background universe. However, Russ et al. (1997) corrected several errors of Bildhauer and Futamase and found that the influence of the inhomogeneities is very small, making the age of the universe less than 10^{-3} smaller than inferred (and smaller than expected implies inhomogeneities actually lead to deceleration, not acceleration of the expansion). Nambu (2000, 2002) found that with spatial averaging inhomogeneities can either accelerate or decelerate the universe's expansion. Kozaki & Nakao (2002) explored Lemaître-Tolman-Bondi models that either contract onto a central singularity or expand to form a dense shell of matter, and found that in both instances the expansion of the universe is decreased.

Since approximation techniques are used to perform spatial averaging, the effect of the inhomogeneities appears to vary drastically depending on the methods and order of approximation used. Also, that the acceleration found by Bildhauer and Futamase only results when structure formation is allowed to occur in one dimension and that the expansion is actually slowed in the direction of structure formation suggests the other dimensions wouldn't be able to compensate by accelerating if structure formation occurred in all dimensions, so it seems unlikely their spatial averaging could be used to explain the acceleration of the universe. Considering Raychaudhuri's equation, inhomogeneities would most simply be expected to introduce shear, which would decrease the volume expansion, and this would be the result even in Newtonian gravity (since tidal forces would still introduce shear and diminish the volume). The spatial averaging schemes all assume no vorticity, so it wouldn't be expected that the volume expansion could increase, unless the relativistic effect of spatial averaging is more significant than the non-relativistic effect of shear, which seems unlikely.

In Chapter 6 the influence of cosmological black holes on the expansion of the universe will be evaluated. Since Weyl curvature represents the relativistic equivalent of tidal forces, calculating the Weyl curvature should reveal whether cosmological black holes would tend to shear the expansion of the universe. The volume expansion will also be calculated, and finally the velocity field, shear, and acceleration will be determined to see whether shear exists and slows the volume expansion of the universe.

2.5 Computer Algebra

The calculation of the metric connections, Einstein's Equations, the Weyl curvature, etc. isn't a trivial task. These calculations will be performed in this thesis using the `REDUCE 3.5` computer algebra program (Hearn 1993) along with the `Redten 4.1e` package (Harper & Dyer 1994) for calculating relativistic quantities. Without the use of computer algebra, many of these quantities would take too long to be humanly possible to determine, especially without error.

It should be noted that while advances in computing have led many into the field of numerical relativity, those same advances have also made it possible to calculate more complicated exact solutions using computer algebra. Many researchers have abandoned the search for exact solutions, believing that no exact solutions remain to be found, or at least no solutions bearing any resemblance to physical reality. However, this belief is fictitious, as computer algebra makes it possible to determine exact solutions for more complicated scenarios, which need not be physically bizarre.

Chapter 3

Kerr-Schild Cosmological Black Holes

In this chapter interpretations will be provided for new Kerr-Schild cosmological Schwarzschild and Reissner-Nordström black holes (and white holes) that are either comoving with the universe's expansion or not and that exist in both radiation-dominated and matter-dominated Einstein-de Sitter universes. Previously, the Schwarzschild white hole comoving with the expansion of a matter-dominated universe was interpreted by Sultana (2003). All of the other solutions presented here are new.

3.1 Radiation-Dominated Universes

A radiation-dominated Einstein-de Sitter universe

$$\mathrm{d}s^2 = t^2 \left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2)\right) \tag{3.1}$$

(with scale factor $R = H_o t$ expressed as $R = t$ for simplicity) has non-zero Einstein tensor components

$$\begin{aligned} G^0_0 &= \frac{3}{t^4} \\ G^1_1 = G^2_2 = G^3_3 &= -\frac{1}{t^4}, \end{aligned} \tag{3.2}$$

which represent the energy density and the negative of the pressure of the radiation as both fall off with $R^4 = t^4$ (where the cosmological time t_c goes as t^2, consistent with R going as ${t_c}^{1/2}$).

3.1.1 Schwarzschild Black Holes

A time-dependent Kerr-Schild transformation (Equation 2.7) of the Einstein-de Sitter universe (Equation 2.5)

$$\mathrm{d}s^2 = t^2\left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + \frac{2m}{r}(\mathrm{d}t \pm \mathrm{d}r)^2\right) \tag{3.3}$$

yields an Einstein-de Sitter-like universe with a Schwarzschild-like black hole (or white hole, depending on the null vector field). Since this metric is equivalent to a conformal transformation of a Schwarzschild black hole, the causal structure is preserved and the event horizon remains at $r = 2m$, but since the r co-ordinate gets scaled by the scale factor, this means the black hole expands with the universe. In co-ordinates comoving with the expansion, the effective mass of the black hole will remain constant, while in non-comoving co-ordinates, the effective mass of the black hole will appear to increase. The transformed metric has non-zero Einstein tensor components

$$\begin{aligned} G^0_0 &= \frac{3}{t^4} \mp \frac{4m}{r^2t^3} + \frac{6m}{rt^4} \\ G^0_1 &= \frac{2m}{r^2t^3} \\ G^1_0 &= -\frac{2m}{r^2t^3} \mp \frac{8m}{rt^4} \\ G^1_1 &= -\frac{1}{t^4} \mp \frac{8m}{r^2t^3} - \frac{2m}{rt^4} \\ G^2_2 = G^3_3 &= -\frac{1}{t^4} - \frac{2m}{rt^4}. \end{aligned} \tag{3.4}$$

Searching for a solution that consists of a perfect fluid component and a heat-conducting component, the Einstein tensor will be given by

$$G^a_b = -\kappa T^a_b = -\kappa\left((\mu + p)u^a u_b + p\delta^a_b + q^a u_b + u^a q_b\right), \tag{3.5}$$

where μ is the energy density, p is the pressure, and q^a is the heat flow vector. Since the solution should be spherically symmetric, then $u^2 = u^3 = 0$, and since $G^0_2 = G^2_0 = G^0_3 = G^3_0 = 0$, then there should be no q^2, q_2, q^3, or q_3 heat conduction components, so that $G^2_2 = G^3_3$ represents only pressure. Since $u^0u_1 = u^1u_0 = 0$, G^1_0 and G^0_1 represent only heat conduction, so the Einstein tensor should correspond to

$$G^0_0 = -\kappa\left((\mu + p)u^0u_0 + p + q^0u_0 + u^0q_0\right)$$

$$
\begin{aligned}
G^0_1 &= -\kappa(q^0 u_1 + u^0 q_1)\\
G^1_0 &= -\kappa(q^1 u_0 + u^1 q_0)\\
G^1_1 &= -\kappa\left((\mu + p)u^1 u_1 + p + q^1 u_1 + u^1 q_1\right)\\
G^2_2 &= G^3_3 = -\kappa p.
\end{aligned} \tag{3.6}
$$

Since $u^a q_a = q^a u_a = 0$ for heat conduction and $u^a u_a = -1$, taking the traces of both of the above forms of the Einstein tensor and comparing them yields

$$G^a_a = \kappa(\mu - 3p) = \mp\frac{12m}{r^2 t^3}, \tag{3.7}$$

which gives the equation of state

$$p = \frac{1}{3}\mu \pm \frac{4m}{\kappa r^2 t^3}. \tag{3.8}$$

Since $G^2_2 = G^3_3 = -\kappa p$,

$$p = \frac{1}{\kappa t^4}\left(1 + \frac{2m}{r}\right), \tag{3.9}$$

so the pressure is in fact always positive. Knowing the equation of state and expression for the pressure, then

$$\mu = 3p \mp \frac{12m}{\kappa r^2 t^3} = \frac{3}{\kappa t^4}\left(1 + \frac{2m}{r} \mp \frac{4mt}{r^2}\right), \tag{3.10}$$

so in the limit as r goes to infinity (or t goes to zero), the energy density is simply three times the pressure, as expected for a radiation-dominated universe.

The energy density will only be positive everywhere in the white hole case. For the black hole, the energy density will become negative and the solution will not be valid for

$$t > \frac{r}{2}\left(1 + \frac{r}{2m}\right). \tag{3.11}$$

Since an ever-increasing region of spacetime becomes invalid at larger times, it is important to know whether any exterior regions causally related to interior regions are actually physical; otherwise, the solution would not be very meaningful if the universe were only able to have a black hole in it only so long as the universe were uninfluenced by it. For the critical case where the energy density is zero

$$\mathrm{d}t = \left(\frac{1}{2} + \frac{r}{2m}\right)\mathrm{d}r. \tag{3.12}$$

The critical curve (independent of θ and ϕ) will be null when

$$0 = g_{00}\mathrm{d}t^2 + 2g_{01}\mathrm{d}t\mathrm{d}r + g_{11}\mathrm{d}r^2 \tag{3.13}$$
$$0 = g_{00}\left(\frac{1}{2}+\frac{r}{2m}\right)^2 \mathrm{d}r^2 + 2g_{01}\left(\frac{1}{2}+\frac{r}{2m}\right)\mathrm{d}r^2 + g_{11}\mathrm{d}r^2$$
$$0 = \left(-1+\frac{2m}{r}\right)\left(\frac{1}{2}+\frac{r}{2m}\right)^2 + \frac{4m}{r}\left(\frac{1}{2}+\frac{r}{2m}\right) + \left(1+\frac{2m}{r}\right)$$
$$0 = \frac{15}{4} + \frac{9m}{2r} - \frac{r^2}{4m^2}.$$

The only positive zero is for

$$\frac{r}{2m} = \frac{\sqrt{33}+3}{4} \doteq 2.186. \tag{3.14}$$

Outside of $r \doteq 4.37m$, the critical curve will be timelike, so it becomes possible for inner regions with positive energy density to be causally related to outer regions with positive energy density.

Since the pressure isn't just one third of the energy density, the negative contribution to the energy density term will cause the pressure to become greater in magnitude than the energy density and violate the dominant energy condition even before the energy density becomes negative. Solving for when the pressure becomes greater than the energy density as above (Equation 3.13), this occurs within

$$\frac{r}{2m} = (\sqrt{3}+1) \doteq 2.732. \tag{3.15}$$

Thus, outside this radius, it becomes possible for inner regions of physical mass-energy to be causally related to outer regions with physical mass-energy, so the solution does have regions of physical interest.

A time-independent Kerr-Schild transformation

$$\mathrm{d}s^2 = t^2\left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right) + \frac{2m}{r}(\mathrm{d}t \pm \mathrm{d}r)^2 \tag{3.16}$$

yields an Einstein-de Sitter-like universe with a Schwarzschild-like black hole that doesn't expand with the rest of the universe.

Vaidya (1977) claimed an event horizon existed when $g^{11} = 0$ for the closed-FRW versions of these black holes, which would suggest the existence of an event horizon at $r = 2mt^{-2}$. However, the radial null curves satisfy

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{t^2 - 2m/r}{t^2 + 2m/r}, \tag{3.17}$$

which suggests that while photons are instantaneously motionless in r at $r = 2m/t^2$, the radius of the surface where $\mathrm{d}r/\mathrm{d}t = 0$ is shrinking to smaller r with time, so that outgoing photons should only momentarily be held at fixed r and not remain trapped as $r = 2m/t^2$ shrinks inward with time. By calculating the normal vector to the surface given by $r = 2m/t^2$, it can be verified that it is null only for $4m = t^3$, which can not remain null for all time. Thus, while this surface implies the inhomogeneity appears to decrease in mass and shrink in radius in time (in both comoving co-ordinates and an observer's non-comoving co-ordinates), it does not appear to be an event horizon. Thus, the definition $g^{11} = 0$ for an event horizon does not appear to be adequate when the radius is not constant with time, so it is unclear whether this spacetime and the Vaidya spacetime actually represent black holes.

The transformed metric has non-zero Einstein tensor components

$$\begin{aligned} G^0_0 &= \frac{3}{t^4} \mp \frac{4m}{r^2 t^5} + \frac{2m}{rt^6} \\ G^0_1 &= -\frac{2m}{r^2 t^5} \\ G^1_0 &= \frac{2m}{r^2 t^5} \mp \frac{8m}{rt^6} \\ G^1_1 &= -\frac{1}{t^4} - \frac{6m}{rt^6} \\ G^2_2 = G^3_3 &= -\frac{1}{t^4} - \frac{4m}{rt^6}. \end{aligned} \tag{3.18}$$

Looking for a solution that consists of a perfect fluid plus heat conduction (Equation 3.5), then once again, taking the trace of the Einstein tensor reveals

$$G^a_a = \kappa(\mu - 3p) = \mp\frac{4m}{r^2 t^5} - \frac{12m}{rt^6}, \tag{3.19}$$

which gives the equation of state

$$p = \frac{1}{3}\mu \pm \frac{4m}{3\kappa r^2 t^5} + \frac{4m}{\kappa r t^6}. \tag{3.20}$$

Since $G^2_2 = G^3_3 = -\kappa p$,

$$p = \frac{1}{\kappa t^4}\left(1 + \frac{4m}{rt^2}\right), \tag{3.21}$$

so the pressure is always positive. Knowing the equation of state and expression for the pressure, then

$$\mu = 3p \mp \frac{4m}{\kappa r^2 t^5} - \frac{12m}{\kappa r t^6} = \frac{3}{\kappa t^4}\left(1 + \frac{4m}{rt^2}\right) \mp \frac{4m}{\kappa r^2 t^5} - \frac{12m}{\kappa r t^6} = \frac{1}{\kappa t^4}\left(3 \mp \frac{4m}{r^2 t}\right), \tag{3.22}$$

so in the limit as r goes to infinity (or t goes to infinity), the energy density is simply three times the pressure, as expected for a radiation-dominated universe. In the white hole case, the energy density is positive everywhere. For the black hole case, the energy density will become negative and the weak energy condition will be violated for

$$t < \frac{4m}{3r^2}, \tag{3.23}$$

which unlike the case of the expanding black hole is less of a problem at large times, but creates problems in the past for ever-increasing values of r as t goes to zero and the Big Bang is approached. As before with the expanding black holes, the requirement that the pressure be smaller in magnitude than the energy density will cause the dominant energy condition to be violated even before the energy density goes negative.

3.1.2 Reissner-Nordström Black Holes

The Reissner-Nordström metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + \left(\frac{2m}{r} - \frac{e^2}{r^2}\right)(\mathrm{d}t \pm \mathrm{d}r)^2 \tag{3.24}$$

has non-zero Einstein tensor components

$$G^0_0 = G^1_1 = -G^2_2 = -G^3_3 = \frac{e^2}{r^4}, \tag{3.25}$$

which stem from the electromagnetic field tensor F^{ab} (which just contains radial electric e/r^2 terms).

A time-dependent Kerr-Schild transformation of the Einstein-de Sitter universe

$$\mathrm{d}s^2 = t^2\left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + \left(\frac{2m}{r} - \frac{e^2}{r^2}\right)(\mathrm{d}t \pm \mathrm{d}r)^2\right) \tag{3.26}$$

yields an Einstein-de Sitter-like universe with a Reissner-Nordström-like black hole that expands with the rest of the universe. The transformed metric has non-zero Einstein tensor components

$$G^0_0 = \frac{3}{t^4} \mp \frac{4m}{r^2t^3} + \frac{6m}{rt^4} + \frac{e^2}{r^4t^2} - \frac{3e^2}{r^2t^4}$$

$$
\begin{aligned}
G^0_1 &= \frac{2m}{r^2t^3} - \frac{2e^2}{r^3t^3} \\
G^1_0 &= -\frac{2m}{r^2t^3} \mp \frac{8m}{rt^4} + \frac{2e^2}{r^3t^3} \pm \frac{4e^2}{r^2t^4} \\
G^1_1 &= -\frac{1}{t^4} \mp \frac{8m}{r^2t^3} - \frac{2m}{rt^4} + \frac{e^2}{r^4t^2} \pm \frac{4e^2}{r^3t^3} + \frac{e^2}{r^2t^4} \\
G^2_2 = G^3_3 &= -\frac{1}{t^4} - \frac{2m}{rt^4} - \frac{e^2}{r^4t^2} \mp \frac{2e^2}{r^3t^3} + \frac{e^2}{r^2t^4}.
\end{aligned} \tag{3.27}
$$

Searching for a solution that consists of a perfect fluid component, a heat-conducting component, and an electromagnetic field component, the Einstein tensor will be given by

$$G^a_b = -\kappa\left((\mu+p)u^a u_b + p\delta^a_b + q^a u_b + u^a q_b + \frac{1}{4\pi}\left(F^{am}F_{mb} + \frac{1}{4}\delta^a_b F_{mn}F^{mn}\right)\right). \tag{3.28}$$

Taking the trace of the electromagnetic field component yields

$$\frac{1}{4\pi}\left(F^{am}F_{ma} + \frac{1}{4}\delta^a_a F_{mn}F^{mn}\right) = \frac{1}{4\pi}\left(-F^{ma}F_{ma} + F_{mn}F^{mn}\right) = 0 \tag{3.29}$$

since the electromagnetic tensor is always antisymmetric. Thus, taking the trace of the Einstein tensor yields

$$G^a_a = \kappa(\mu - 3p) = \mp\frac{12m}{r^2t^3}. \tag{3.30}$$

Looking at $G^2_2 = G^3_3$, the $e^2/(r^4t^2)$ terms are apparently the time-dependent version of the electric terms that appear in the source for the standard Reissner-Nordström spacetime, while the $1/t^4$, $2m/(rt^4)$, and $e^2/(r^2t^4)$ terms stand out as being $-1/3$ of the corresponding terms in G^0_0, suggesting these are pressure terms since normally $G^0_0 = \kappa\mu$ and $G^1_1 = G^2_2 = G^3_3 = -\kappa p = -\kappa\mu/3$ for a radiation-dominated universe. Since it isn't possible to modify the electromagnetic tensor to generate the $e^2/(r^3t^3)$ terms, and since the term in G^1_1 cancels the terms in $G^2_2 = G^3_3$, these terms appear to modify the average pressure in the radial and angular directions such that the average pressure remains unchanged. Thus, the average pressure and energy density should be given by

$$p = \frac{1}{\kappa t^4}\left(1 + \frac{2m}{r} - \frac{e^2}{r^2}\right) = \frac{1}{3}\mu \pm \frac{4m}{\kappa r^2 t^3} \tag{3.31}$$

and

$$\mu = 3p \mp \frac{12m}{\kappa r^2 t^3} = \frac{3}{\kappa t^4}\left(1 + \frac{2m}{r} - \frac{e^2}{r^2} \mp \frac{4mt}{r^2}\right), \tag{3.32}$$

analogous to the Schwarzschild case (cf. Equations 3.9 and 3.10). The electric field can now make the pressure and energy density go negative; however, the highest charge-to-mass ratio that exists is that of an electron, for which the magnitude of the electric term is only greater than that of the mass term for $r < 1.41 \text{ x } 10^{-15}$ m, which is also the scale where the Strong force should come into play (and perhaps also quantum mechanics), so the magnitude of the electric term should always be less than that of the mass term except at scales where the physics isn't expected to be accurately represented in the first place. As with the uncharged black hole, the negative term in the expression for the energy density can cause the energy density to go negative and violate the weak energy condition, as well as allow the pressure to become greater in magnitude than the energy density and violate the dominant energy condition.

A time-independent Kerr-Schild transformation

$$\mathrm{d}s^2 = t^2\left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right) + \left(\frac{2m}{r} - \frac{e^2}{r^2}\right)(\mathrm{d}t \pm \mathrm{d}r)^2 \tag{3.33}$$

yields an Einstein-de Sitter-like universe with a Reissner-Nordström-like black hole that doesn't expand with the rest of the universe (once again, it should be noted that since the black hole is shrinking, there may not be a true event horizon, so it might not actually be a black hole in the ordinary sense). The transformed metric has non-zero Einstein tensor components

$$\begin{aligned}
G^0_0 &= \frac{3}{t^4} \mp \frac{4m}{r^2t^5} + \frac{2m}{rt^6} + \frac{e^2}{r^4t^4} - \frac{e^2}{r^2t^6} \\
G^0_1 &= -\frac{2m}{r^2t^5} \\
G^1_0 &= \frac{2m}{r^2t^5} \mp \frac{8m}{rt^6} \pm \frac{4e^2}{r^2t^6} \\
G^1_1 &= -\frac{1}{t^4} - \frac{6m}{rt^6} + \frac{e^2}{r^4t^4} + \frac{3e^2}{r^2t^6} \\
G^2_2 = G^3_3 &= -\frac{1}{t^4} - \frac{4m}{rt^6} - \frac{e^2}{r^4t^4} + \frac{2e^2}{r^2t^6}.
\end{aligned} \tag{3.34}$$

Once again, taking the trace of the Einstein tensor (Equation 3.28) reveals

$$G^a_a = \kappa(\mu - 3p) = \mp\frac{4m}{r^2 t^5} - \frac{12m}{rt^6} + \frac{6e^2}{r^2 t^6}, \tag{3.35}$$

which gives the pressure and energy density as

$$p = \frac{1}{\kappa t^4}\left(1 + \frac{4m}{rt^2} - \frac{2e^2}{r^2 t^2}\right) = \frac{1}{3}\mu \pm \frac{4m}{3\kappa r^2 t^5} + \frac{4m}{\kappa r t^6} - \frac{2e^2}{\kappa r^2 t^6} \tag{3.36}$$

and

$$\mu = 3p \mp \frac{4m}{\kappa r^2 t^5} - \frac{12m}{\kappa r t^6} + \frac{6e^2}{\kappa r^2 t^6} = \frac{1}{\kappa t^4}\left(3 \mp \frac{4m}{r^2 t}\right), \tag{3.37}$$

analogous to the corresponding Schwarzschild case (cf. Equations 3.21 and 3.22). Interestingly, the charge decreases the pressure such that the energy density is unmodified by the charge.

3.2 Matter-Dominated Universes

A matter-dominated Einstein-de Sitter universe

$$\mathrm{d}s^2 = t^4\left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right) \tag{3.38}$$

(with scale factor $R = (H_o t/2)^2$ expressed as $R = t^2$ for simplicity) has one non-zero Einstein tensor component

$$G^0_0 = \frac{12}{t^6}, \tag{3.39}$$

which represents the energy density as it falls off with $R^3 = t^6$ (where the cosmological time t_c goes as t^3, consistent with R going as ${t_c}^{2/3}$).

3.2.1 Schwarzschild Black Holes

A time-dependent Kerr-Schild transformation of the Einstein-de Sitter universe

$$\mathrm{d}s^2 = t^4\left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + \frac{2m}{r}(\mathrm{d}t \pm \mathrm{d}r)^2\right) \tag{3.40}$$

yields an Einstein-de Sitter-like universe with a Schwarzschild-like feature that expands with the rest of the universe. The white hole case was previously interpreted by Sultana (2003). The transformed metric has non-zero Einstein tensor components

$$G^0_0 = \frac{12}{t^6} \mp \frac{8m}{r^2 t^5} + \frac{24m}{rt^6}$$

$$G^0_1 = \frac{4m}{r^2t^5}$$
$$G^1_0 = -\frac{4m}{r^2t^5} \mp \frac{24m}{rt^6}$$
$$G^1_1 = \mp\frac{16m}{r^2t^5}. \tag{3.41}$$

Looking for a solution that consists of a pressureless dust (since $G^2_2 = G^3_3 = 0$ suggests $p = 0$, as would be expected for a matter-dominated universe) with heat conduction, analogous to the radiation-dominated case (Equation 3.5), then taking the trace of the Einstein tensor yields

$$G^a_a = \kappa\mu = \frac{12}{t^6} \mp \frac{24m}{r^2t^5} + \frac{24m}{rt^6}. \tag{3.42}$$

The energy density is

$$\mu = \frac{12}{\kappa t^6}\left(1 + \frac{2m}{r} \mp \frac{2mt}{r^2}\right). \tag{3.43}$$

In the limit as r goes to infinity, the energy density becomes identical to that of the plain Einstein-de Sitter universe. For the white hole case, the energy density will be positive everywhere (as previously shown by Sultana 2003). For the black hole case, the energy density will become negative and violate the weak energy condition, so the solution will be invalid for

$$t > r + \frac{r^2}{2m}. \tag{3.44}$$

Performing the calculation as in the radiation-dominated case (Equation 3.13), the critical curve for zero energy density is null when

$$0 = 8 + \frac{8m}{r} - \frac{r^2}{m^2}, \tag{3.45}$$

which has a positive zero only for

$$\frac{r}{2m} = \frac{\sqrt{5}+1}{2} \doteq 1.618 \tag{3.46}$$

(which interestingly is the Golden Ratio). Outside of $r \doteq 3.24m$, the critical curve will be timelike, so it becomes possible for inner regions with physical energy density to be causally related to outer regions with physical energy density.

A time-independent Kerr-Schild transformation

$$\mathrm{d}s^2 = t^4\left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right) + \frac{2m}{r}(\mathrm{d}t \pm \mathrm{d}r)^2 \tag{3.47}$$

yields an Einstein-de Sitter-like universe with a Schwarzschild-like black hole that doesn't expand with the rest of the universe. The transformed metric has non-zero Einstein tensor components

$$\begin{aligned} G^0_0 &= \frac{12}{t^6} \mp \frac{8m}{r^2 t^9} + \frac{8m}{r t^{10}} \\ G^0_1 &= -\frac{4m}{r^2 t^9} \\ G^1_0 &= \frac{4m}{r^2 t^9} \mp \frac{24m}{r t^{10}} \\ G^1_1 &= -\frac{16m}{r t^{10}} \\ G^2_2 = G^3_3 &= -\frac{12m}{r t^{10}}. \end{aligned} \tag{3.48}$$

Looking for a perfect fluid (since $G^2_2 = G^3_3 \neq 0$ suggests the pressure is only asymptotically zero as r goes to infinity) plus heat conduction source (Equation 3.5), then once again, taking the trace of the Einstein tensor yields

$$G^a_a = \kappa(\mu - 3p) = \frac{12}{t^6} \mp \frac{8m}{r^2 t^9} - \frac{32m}{r t^{10}}. \tag{3.49}$$

Looking at $G^2_2 = G^3_3$, assuming the $m/(rt^{10})$ terms represent a modification to the normally zero pressure of a matter-dominated universe, then the pressure and energy density will be given by

$$p = \frac{12m}{\kappa r t^{10}} = \frac{1}{3}\mu - \frac{4}{\kappa t^6} \pm \frac{8m}{3\kappa r^2 t^9} + \frac{32m}{3\kappa r t^{10}} \tag{3.50}$$

and

$$\mu = 3p + \frac{12}{\kappa t^6} \mp \frac{8m}{\kappa r^2 t^9} - \frac{32m}{\kappa r t^{10}} = \frac{4}{\kappa t^6}\left(3 + \frac{m}{r t^4} \mp \frac{2m}{r^2 t^3}\right). \tag{3.51}$$

In the limit as r goes to infinity (or t goes to infinity) the pressure and energy density approach that of the plain Einstein-de Sitter universe. In the black hole case, the energy density will become negative and violate the weak energy condition for

$$t > \frac{r}{2} + \frac{3r^2 t^4}{2m}, \tag{3.52}$$

which is problematic at small values of r, but unlike the radiation-dominated case (cf. Equation 3.23) doesn't extend to all r as t goes to zero and the Big Bang is approached. Since the pressure isn't zero, the dominant energy condition will also be violated even before the energy density becomes negative.

3.2.2 Reissner-Nordström Black Holes

A time-dependent Kerr-Schild transformation of the Einstein-de Sitter universe

$$\mathrm{d}s^2 = t^4 \left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2) + \left(\frac{2m}{r} - \frac{e^2}{r^2} \right) (\mathrm{d}t \pm \mathrm{d}r)^2 \right) \tag{3.53}$$

yields an Einstein-de Sitter-like universe with a Reissner-Nordström-like black hole that expands with the rest of the universe. The transformed metric has non-zero Einstein tensor components

$$\begin{aligned} G^0_0 &= \frac{12}{t^6} \mp \frac{8m}{r^2 t^5} + \frac{24m}{r t^6} + \frac{e^2}{r^4 t^4} - \frac{12e^2}{r^2 t^6} \\ G^0_1 &= \frac{4m}{r^2 t^5} - \frac{4e^2}{r^3 t^5} \\ G^1_0 &= -\frac{4m}{r^2 t^5} \mp \frac{24m}{r t^6} \pm \frac{12e^2}{r^2 t^6} + \frac{4e^2}{r^3 t^5} \\ G^1_1 &= \mp\frac{16m}{r^2 t^5} + \frac{e^2}{r^4 t^4} \pm \frac{8e^2}{r^3 t^5} \\ G^2_2 = G^3_3 &= -\frac{e^2}{r^4 t^4} \mp \frac{4e^2}{r^3 t^5}. \end{aligned} \tag{3.54}$$

Looking for a solution that consists of a dust with heat conduction plus electric field, analogous to the radiation-dominated case (Equation 3.28) taking the trace of the Einstein tensor yields

$$G^a_a = \kappa\mu = \frac{12}{t^6} \mp \frac{24m}{r^2 t^5} + \frac{24m}{r t^6} - \frac{12e^2}{r^2 t^6}. \tag{3.55}$$

Looking at $G^2_2 = G^3_3$, the $e^2/(r^4 t^4)$ terms are apparently the time-dependent versions of the electric terms that appear in the source for the standard Reissner-Nordström spacetime, while the $e^2/(r^3 t^5)$ terms cancel the corresponding term in G^1_1 and appear to modify the average pressure from zero in the radial and angular directions (such that the average pressure remains zero). Thus, the average pressure is zero and the energy density is given by

$$\mu = \frac{12}{\kappa t^6} \left(1 + \frac{2m}{r} - \frac{e^2}{r^2} \mp \frac{2mt}{r^2} \right), \tag{3.56}$$

analogous to the corresponding Schwarzschild case (cf. Equation 3.43). As discussed previously for Reissner-Nordström black holes in radiation-dominated Einstein-de Sitter universes, the mass term should dominate the electric term to prevent the electric term from making the energy density become negative.

A time-independent Kerr-Schild transformation

$$\mathrm{d}s^2 = t^4\left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right) + \left(\frac{2m}{r} - \frac{e^2}{r^2}\right)(\mathrm{d}t \pm \mathrm{d}r)^2 \tag{3.57}$$

yields an Einstein-de Sitter-like universe with a Reissner-Nordström-like black hole that doesn't expand with the rest of the universe. The transformed metric has non-zero Einstein tensor components

$$\begin{aligned} G^0_0 &= \frac{12}{t^6} \mp \frac{8m}{r^2t^9} + \frac{8m}{rt^{10}} + \frac{e^2}{r^4t^8} - \frac{4e^2}{r^2t^{10}} \\ G^0_1 &= -\frac{4m}{r^2t^9} \\ G^1_0 &= \frac{4m}{r^2t^9} \mp \frac{24m}{rt^{10}} \pm \frac{12e^2}{r^2t^{10}} \\ G^1_1 &= -\frac{16m}{rt^{10}} + \frac{e^2}{r^4t^8} + \frac{8e^2}{r^2t^{10}} \\ G^2_2 = G^3_3 &= -\frac{12m}{rt^{10}} - \frac{e^2}{r^4t^8} + \frac{6e^2}{r^2t^{10}}. \end{aligned} \tag{3.58}$$

Looking for a perfect fluid (since $G^2_2 = G^3_3 \neq 0$ suggests that the black hole modifies the pressure from zero except asymptotically as r goes to infinity) with heat conduction plus electric field source (Equation 3.28), taking the trace of the Einstein tensor yields

$$G^a_a = \kappa(\mu - 3p) = \frac{12}{t^6} \mp \frac{8m}{r^2t^9} - \frac{32m}{rt^{10}} + \frac{16e^2}{r^2t^{10}}. \tag{3.59}$$

Looking at $G^2_2 = G^3_3$ as before, the $e^2/(r^4t^8)$ terms are apparently the time-dependent versions of the electric terms that appear in the source for the standard Reissner-Nordström spacetime, so assuming the $m/(rt^{10})$ and $e^2/(r^2t^{10})$ terms represent a modification to the normally zero pressure of a matter-dominated universe, then the pressure and energy density will be given by

$$p = \frac{6}{\kappa t^{10}}\left(\frac{2m}{r} - \frac{e^2}{r^2}\right) = \frac{1}{3}\mu - \frac{4}{\kappa t^6} \pm \frac{8m}{3\kappa r^2t^9} + \frac{32m}{3\kappa rt^{10}} - \frac{16e^2}{3\kappa r^2t^{10}} \tag{3.60}$$

and

$$\mu = 3p + \frac{12}{\kappa t^6} \mp \frac{8m}{\kappa r^2 t^9} - \frac{32m}{\kappa r t^{10}} + \frac{16e^2}{\kappa r^2 t^{10}} = \frac{2}{\kappa t^6}\left(6 + \frac{2m}{rt^4} - \frac{e^2}{r^2 t^4} \mp \frac{4m}{r^2 t^3}\right), \quad (3.61)$$

analogous to the corresponding Schwarzschild case (cf. Equations 3.50 and 3.51).

Chapter 4

Isotropic Cosmological Black Holes

In this chapter interpretations will be provided for new isotropic cosmological black hole solutions that will be obtained by performing conformal transformations on isotropic black hole spacetimes. The M[c]Vittie cosmological black hole spacetimes will also be reinterpreted to determine where they are physical to see where they are valid solutions of the Field Equations, since this does not appear to have been done previously. The Schwarzschild cases will be examined in the first section, and the Reissner-Nordström cases will be examined in the second section.

4.1 Schwarzschild Black Holes

4.1.1 Conformal Transformation of an Isotropic Black Hole

In this section, a metric analogous to the Thakurta (1981) metric or Sultana (2003) metric (e.g. Equation 3.40) will be obtained by starting with a Schwarzschild black hole in isotropic co-ordinates prior to performing the conformal transformation.

A Schwarzschild black hole

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{\bar{r}}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{\bar{r}}\right)^{-1}\mathrm{d}\bar{r}^2 + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2) \tag{4.1}$$

can be written in isotropic co-ordinates

$$\mathrm{d}s^2 = -\left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2\mathrm{d}t^2 + \left(1 + \frac{m}{2r}\right)^4(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2)) \tag{4.2}$$

via a transformation of the areal co-ordinate $\bar{r}$

$$\bar{r} = r\left(1+\frac{m}{2r}\right)^2, \tag{4.3}$$

such that the event horizon $\bar{r} = 2m$ is at $r = m/2$ in the new radial co-ordinate. Performing a conformal transformation to obtain a cosmological black hole

$$\mathrm{d}s^2 = [R(t)]^2\left(-\left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^2 \mathrm{d}t^2 + \left(1+\frac{m}{2r}\right)^4\left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2)\right)\right) \tag{4.4}$$

and then making the transformation $\mathrm{d}t_c = R(t)\mathrm{d}t$, the following metric is obtained, which in the limit as r goes to infinity looks like the standard form of the Robertson-Walker metric for an Einstein-de Sitter universe,

$$\mathrm{d}s^2 = -\left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^2 \mathrm{d}{t_c}^2 + [R(t_c)]^2\left(1+\frac{m}{2r}\right)^4\left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2)\right). \tag{4.5}$$

The event horizon remains at $r = m/2$, since the conformal transformation preserves the causal structure of the original isotropic black hole spacetime. The expansion of the universe scales the r dimension such that objects comoving with the expansion (remaining at fixed r) have their spatial separation increase with R, so the event horizon of the black hole grows with the expansion of the universe such that it appears to remain fixed in size in co-ordinates comoving with the universe's expansion. Thus, this metric appears to represent a cosmological black hole that expands with the universe, with the effective mass of the black hole appearing to remain the same in comoving co-ordinates or grow according to the expansion of the universe in non-comoving co-ordinates.

This metric has non-zero Einstein tensor components

$$G^0_0 = \frac{3(m+2r)^2\dot{R}^2}{(m-2r)^2R^2}$$

$$G^0_1 = -\frac{8m(m+2r)\dot{R}}{(m-2r)^3R}$$

$$G^1_0 = \frac{128mr^4\dot{R}}{(m+2r)^5(m-2r)R^3}$$

$$G^1_1 = G^2_2 = G^3_3 = \frac{\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right)(m+2r)^2}{(m-2r)^2}. \tag{4.6}$$

Looking for a solution that consists of a perfect fluid plus a heat conduction component, the Einstein tensor (Equation 3.5) should correspond to

$$\begin{aligned}
G^0_0 &= -\kappa\left((\mu+p)u^0u_0 + p + q^0u_0 + u^0q_0\right)\\
G^0_1 &= -\kappa(q^0u_1 + u^0q_1)\\
G^1_0 &= -\kappa(q^1u_0 + u^1q_0)\\
G^1_1 &= -\kappa\left((\mu+p)u^1u_1 + p + q^1u_1 + u^1q_1\right)\\
G^2_2 &= G^3_3 = -\kappa p.
\end{aligned} \tag{4.7}$$

Since $u^aq_a = q^au_a = 0$ for heat conduction and $u^au_a = -1$, taking the traces of both of the above Einstein tensors and comparing them yields

$$G^a_a = \kappa(\mu - 3p) = \frac{\left(6\frac{\ddot{R}}{R} + 6\frac{\dot{R}^2}{R^2}\right)(m+2r)^2}{(m-2r)^2}, \tag{4.8}$$

which gives the equation of state

$$p = \frac{1}{3}\mu - \frac{\left(2\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2}\right)(m+2r)^2}{\kappa(m-2r)^2}. \tag{4.9}$$

Since $G^2_2 = G^3_3 = -\kappa p$,

$$p = -\frac{\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right)(m+2r)^2}{\kappa(m-2r)^2}. \tag{4.10}$$

Thus, the energy density is given by

$$\mu = 3p + \frac{\left(6\frac{\ddot{R}}{R} + 6\frac{\dot{R}^2}{R^2}\right)(m+2r)^2}{\kappa(m-2r)^2} = \frac{\left(3\frac{\dot{R}^2}{R^2}\right)(m+2r)^2}{\kappa(m-2r)^2}. \tag{4.11}$$

For a radiation-dominated universe, R goes as ${t_c}^{1/2}$, which yields

$$\mu = \frac{3(m+2r)^2}{\kappa 4{t_c}^2(m-2r)^2} \tag{4.12}$$

and

$$p = \frac{(m+2r)^2}{\kappa 4{t_c}^2(m-2r)^2}. \tag{4.13}$$

Thus, the energy density is always positive and the pressure is one third of the energy density everywhere, just as it is for a radiation-dominated FRW universe, and the energy conditions are satisfied everywhere. In the limit as r goes to zero or r goes to infinity, the energy density and pressure become that of the standard FRW universe. As the event horizon $r = m/2$ is approached from either side, the energy density and pressure both approach infinity.

For a matter-dominated universe, R goes as ${t_c}^{2/3}$, which yields

$$\mu = \frac{4(m+2r)^2}{\kappa 3{t_c}^2(m-2r)^2} \tag{4.14}$$

and $p = 0$. Thus, the energy density is positive everywhere, becoming infinite at the event horizon and approaching the energy density of a matter-dominated FRW universe as r goes to zero or infinity, just as in the radiation-dominated case. The pressure is zero everywhere, just as it is for a matter-dominated FRW universe. Since the energy density is always positive and the pressure is zero, the energy conditions are satisfied everywhere.

4.1.2 McVittie's Point Mass in an Expanding Universe

The metric derived in the previous section is similar to McVittie's (1933) metric, except that McVittie's metric (in the case of an asymptotically flat universe) is

$$\mathrm{d}s^2 = -\left(\frac{1-\frac{m/R}{2r}}{1+\frac{m/R}{2r}}\right)^2 \mathrm{d}{t_c}^2 + [R(t_c)]^2\left(1+\frac{m/R}{2r}\right)^4\left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right). \tag{4.15}$$

so that the mass m/R varies and is scaled down by the expansion of the universe. In co-ordinates comoving with the universe's expansion the surface $r = m/(2R)$ shrinks according to the scale factor as the universe expands, but since the scale factor is scaling the r dimension according to R, then the net effect is that the spatial extent of this surface in non-comoving co-ordinates stays constant ($rR = m/2$).

It should be noted that the radial null curves satisfy

$$R\frac{\mathrm{d}r}{\mathrm{d}t_c} = \pm\frac{1-m/(2rR)}{(1+m/(2rR))^3}, \tag{4.16}$$

which suggests that while photons remain instantaneously motionless in r at $rR = m/2$ (since $\mathrm{d}r/\mathrm{d}t_c = 0$ there), they do not remain motionless in rR since it is not the case that $\mathrm{d}(rR)/\mathrm{d}t_c = 0$ there. It appears that as the radius $r = m/(2R)$ decreases with time, the photons will only momentarily be held at fixed r and will eventually be able to move outward as the surface $r = m/(2R)$ moves inward. Thus, it is not clear that the surface $rR = m/2$ acts as a event horizon, so as with the Vaidya-type black holes the M^{c}Vittie spacetime may not actually be a black hole, despite its resemblance to one.

Looking at the Einstein tensor as in the previous section, the only non-zero components are

$$G_0^0 = \frac{3\dot{R}^2}{R^2}$$

$$G_1^1 = G_2^2 = G_3^3 = \frac{2(2rR+m)\frac{\ddot{R}}{R} + (2rR-5m)\frac{\dot{R}^2}{R^2}}{2rR-m}, \tag{4.17}$$

which yield the energy density and pressure as

$$\mu = \frac{3\dot{R}^2}{\kappa R^2} \tag{4.18}$$

and

$$p = -\frac{2(2rR+m)\frac{\ddot{R}}{R} + (2rR-5m)\frac{\dot{R}^2}{R^2}}{\kappa(2rR-m)} \tag{4.19}$$

with no heat conduction. Thus, the energy density is spatially homogeneous, taking on the usual FRW value, while the pressure is infinite at $r = m/(2R)$ and asymptotically approaches the usual FRW pressure only as r goes to infinity.

In the case of a radiation-dominated background universe, the energy density and pressure will be given by

$$\mu = \frac{3}{4\kappa {t_c}^2} \tag{4.20}$$

and

$$p = \frac{2rR+7m}{4\kappa {t_c}^2(2rR-m)}. \tag{4.21}$$

Thus, it is apparent that inside $r = m/(2R)$, the pressure is negative, ranging from

$$p = -\frac{7}{4\kappa {t_c}^2} \tag{4.22}$$

as r approaches zero, to negative infinity as r approaches $m/(2R)$ from within. Outside $r = m/(2R)$, the pressure falls off from positive infinity at $r = m/(2R)$ to

$$p = \frac{1}{4\kappa {t_c}^2} \tag{4.23}$$

as r approaches infinity. Since the pressure is negative and greater in magnitude than the energy density everywhere inside $r = m/(2R)$, it violates all of the energy conditions there. Outside $r = m/(2R)$, the magnitude of the pressure is greater than that of the energy density for $r < 5m/(2R)$, so the dominant energy condition is also violated in that region.

In the case of a matter-dominated background universe, the energy density and pressure will be given by

$$\mu = \frac{4}{3\kappa {t_c}^2} \tag{4.24}$$

and

$$p = \frac{8m}{3\kappa {t_c}^2(2rR - m)}. \tag{4.25}$$

Thus, it is apparent that inside $r = m/(2R)$, the pressure is negative, ranging from

$$p = -\frac{8}{3\kappa {t_c}^2} \tag{4.26}$$

as r approaches zero, to negative infinity as r approaches $m/(2R)$ from within. Outside $r = m/(2R)$, the pressure falls off from positive infinity at $r = m/(2R)$, falling off as $1/r$ for large r and approaching zero as r goes to infinity. Since the pressure is negative and greater in magnitude than the energy density everywhere inside $r = m/(2R)$, it violates all of the energy conditions there. Outside $r = m/(2R)$, the magnitude of the pressure is greater than that of the energy density for $r < 3m/(2R)$, so the dominant energy condition is also violated in that region.

4.2 Reissner-Nordström Black Holes

4.2.1 Conformal Transformation of an Isotropic Reissner-Nordström Black Hole

The metric for a Reissner-Nordström black hole in isotropic co-ordinates, originally given by Prasanna (1968), is

$$\mathrm{d}s^2 = -\left(\frac{1 - \frac{m^2}{4r^2} + \frac{e^2}{4r^2}}{\left(1 + \frac{m}{2r}\right)^2 - \frac{e^2}{4r^2}}\right)^2 \mathrm{d}t^2 + \left(\left(1 + \frac{m}{2r}\right)^2 - \frac{e^2}{4r^2}\right)^2 (\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)), \tag{4.27}$$

so performing a conformal transformation and then making the transformation $\mathrm{d}t_c = R(t)\mathrm{d}t$ yields

$$\mathrm{d}s^2 = -\left(\frac{1-\frac{m^2}{4r^2}+\frac{e^2}{4r^2}}{\left(1+\frac{m}{2r}\right)^2-\frac{e^2}{4r^2}}\right)^2 \mathrm{d}{t_c}^2 + [R(t_c)]^2\left(\left(1+\frac{m}{2r}\right)^2-\frac{e^2}{4r^2}\right)^2(\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)) \quad (4.28)$$

for a charged black hole that expands along with an asymptotically Einstein-de Sitter universe.

This metric has non-zero Einstein tensor components

$$G^0_0 = \frac{3(e+m+2r)^2(e-m-2r)^2}{(e^2-m^2+4r^2)^2}\frac{\dot{R}^2}{R^2} + \frac{256e^2r^4}{(e+m+2r)^4(e-m-2r)^4R^2}$$

$$G^0_1 = -\frac{8(e^2(m+4r)-m(m+2r)^2)(e+m+2r)(e-m-2r)}{(e^2-m^2+4r^2)^3}\frac{\dot{R}}{R}$$

$$G^1_0 = \frac{128r^4(e^2(m+4r)-m(m+2r)^2)}{(e^2-m^2+4r^2)^2(e+m+2r)^3(e-m-2r)^3}\frac{\dot{R}}{R^3}$$

$$G^1_1 = \frac{(e+m+2r)^2(e-m-2r)^2}{(e^2-m^2+4r^2)^2}\left(2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}\right) + \frac{256e^2r^4}{(e+m+2r)^4(e-m-2r)^4R^2}$$

$$G^2_2 = G^3_3 = \frac{(e+m+2r)^2(e-m-2r)^2}{(e^2-m^2+4r^2)^2}\left(2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}\right) - \frac{256e^2r^4}{(e+m+2r)^4(e-m-2r)^4R^2}. \quad (4.29)$$

Looking for a solution that consists of a perfect fluid plus heat conduction component plus electric field (Equation 3.28), the terms that depend on the scale factor as $1/R^2$ correspond to the usual electric field components of an isolated Reissner-Nordström black hole

$$G^0_0 = G^1_1 = -G^2_2 = -G^3_3 = \frac{256e^2r^4}{(e+m+2r)^4(e-m-2r)^4}, \quad (4.30)$$

so these simply represent the electric field. Thus, the pressure is given by

$$p = -\frac{(e+m+2r)^2(e-m-2r)^2}{\kappa(e^2-m^2+4r^2)^2}\left(2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}\right) \quad (4.31)$$

and the energy density is given by

$$\mu = \frac{3(e+m+2r)^2(e-m-2r)^2}{\kappa(e^2-m^2+4r^2)^2}\frac{\dot{R}^2}{R^2}. \quad (4.32)$$

In the case of a radiation-dominated universe, the pressure and energy density are given by

$$p = \frac{(e+m+2r)^2(e-m-2r)^2}{\kappa(e^2-m^2+4r^2)^2}\frac{1}{4t_c^2} \tag{4.33}$$

and

$$\mu = \frac{(e+m+2r)^2(e-m-2r)^2}{\kappa(e^2-m^2+4r^2)^2}\frac{3}{4t_c^2}, \tag{4.34}$$

and in the case of a matter-dominated universe, the pressure is zero and the energy density is given by

$$\mu = \frac{(e+m+2r)^2(e-m-2r)^2}{\kappa(e^2-m^2+4r^2)^2}\frac{4}{3t_c^2}. \tag{4.35}$$

The energy density is always positive, and in the case of a radiation-dominated universe the pressure is always one third of the energy density, or in the case of a matter-dominated universe the pressure is always zero (just as in the case of the isotropic Schwarzschild cosmological black hole). Thus, the energy conditions are everywhere satisfied by this solution.

4.2.2 Charged M^cVittie Black Holes

Gao & Zhang (2004) generalized M^cVittie's solution to include charge. In the case of a flat universe, the metric is given by

$$\mathrm{d}s^2 = -\left(\frac{1-\frac{m^2/R^2}{4r^2}+\frac{e^2/R^2}{4r^2}}{\left(1+\frac{m/R}{2r}\right)^2-\frac{e^2/R^2}{4r^2}}\right)^2 \mathrm{d}{t_c}^2 + [R(t_c)]^2\left(\left(1+\frac{m/R}{2r}\right)^2-\frac{e^2/R^2}{4r^2}\right)^2(\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)), \tag{4.36}$$

so the mass m/R and charge e/R both vary and get scaled down by the expansion of the universe and the surface $r=\sqrt{m^2-e^2}/(2R)$ shrinks in comoving co-ordinates, although they will all appear constant in non-comoving co-ordinates. As with the M^cVittie spacetime, it should be noted that the surface $r=\sqrt{m^2-e^2}/(2R)$ does not appear to actually be an event horizon.

The non-zero Einstein tensor components are

$$G_0^0 = \frac{3\dot{R}^2}{R^2} + \frac{256e^2r^4R^4}{(e+m+2r)^4(e-m-2r)^4}$$

$$G^1_1 = \frac{2(4r^2R^2 + 4rmR - e^2 + m^2)\frac{\ddot{R}}{R} + (4r^2R^2 - 8mrR + 5e^2 - 5m^2)\frac{\dot{R}^2}{R^2}}{4r^2R^2 + e^2 - m^2} + \frac{256e^2r^4R^4}{(e+m+2r)^4(e-m-2r)^4}$$

$$G^2_2 = G^3_3 = \frac{2(4r^2R^2 + 4rmR - e^2 + m^2)\frac{\ddot{R}}{R} + (4r^2R^2 - 8mrR + 5e^2 - 5m^2)\frac{\dot{R}^2}{R^2}}{4r^2R^2 + e^2 - m^2} - \frac{256e^2r^4R^4}{(e+m+2r)^4(e-m-2r)^4}. \tag{4.37}$$

Looking for a solution that consists of a perfect fluid plus electric field (Equation 3.28 without heat conduction), the terms that go as R^4 clearly correspond to the electric field terms for the isolated isotropic Reissner-Nordström black hole (Equation 4.30), so the energy density and pressure are given by

$$\mu = \frac{3\dot{R}^2}{\kappa R^2} \tag{4.38}$$

and

$$p = -\frac{2(4r^2R^2 + 4rmR - e^2 + m^2)\frac{\ddot{R}}{R} + (4r^2R^2 - 8mrR + 5e^2 - 5m^2)\frac{\dot{R}^2}{R^2}}{\kappa(4r^2R^2 + e^2 - m^2)}. \tag{4.39}$$

In the radiation-dominated case the energy density

$$\mu = \frac{3}{4\kappa t_c^2} \tag{4.40}$$

is spatially uniform, as with the M^cVittie spacetime (cf. Equation 4.20), and the pressure is

$$p = \frac{4r^2R^2 + 16mrR - 7e^2 + 7m^2}{4\kappa(4r^2R^2 + e^2 - m^2)t_c^2}, \tag{4.41}$$

which becomes negative inside $r = \sqrt{m^2 - e^2}/(2R)$ where the denominator changes signs (the numerator can be considered positive since m^2 should dominate e^2). Thus, inside $r = \sqrt{m^2 - e^2}/(2R)$ the pressure goes from

$$p = -\frac{7}{4\kappa {t_c}^2} \tag{4.42}$$

as r approaches zero, to negative infinity as r approaches $\sqrt{m^2 - e^2}/(2R)$ from within. Outside $r = \sqrt{m^2 - e^2}/(2R)$, the pressure falls off from positive infinity at $r = \sqrt{m^2 - e^2}/(2R)$ to

$$p = \frac{1}{4\kappa {t_c}^2} \tag{4.43}$$

as r approaches infinity. Since the pressure is negative and greater in magnitude than the energy density everywhere inside $r = \sqrt{m^2 - e^2}/(2R)$, it violates all of the energy conditions there, and outside $r = \sqrt{m^2 - e^2}/(2R)$, since the magnitude of the pressure will be greater than that of the energy density within some radius, the dominant energy condition is violated in that region.

In the matter-dominated case the energy density is given by

$$\mu = \frac{4}{3\kappa {t_c}^2}, \tag{4.44}$$

which is also uniform (cf. Equation 4.24), and the pressure is given by

$$p = \frac{8(2mrR - e^2 + m^2)}{3\kappa(4r^2R^2 + e^2 - m^2)t_c^2}, \tag{4.45}$$

so the pressure varies from

$$p = -\frac{8}{3\kappa {t_c}^2} \tag{4.46}$$

as r approaches zero, to negative infinity as r approaches $\sqrt{m^2 - e^2}/(2R)$ from within, and from positive infinity as r approaches $\sqrt{m^2 - e^2}/(2R)$ from outside to zero as r approaches infinity. As with the uncharged McVittie black holes, all the energy conditions are violated inside $r = \sqrt{m^2 - e^2}/(2R)$, and the dominant energy condition is violated within some radius beyond $r = \sqrt{m^2 - e^2}/(2R)$.

Chapter 5

Matching Radiation Universes to Dust Universes

In this chapter it will be shown that a radiation-dominated universe can be directly matched onto a matter-dominated universe across a hypersurface of constant time, allowing for the possibility of a universe that instantaneously converts its radiation into matter or converts its matter into radiation. This will then be applied to the Kerr-Schild and isotropic cosmological black hole spacetimes to create simple solutions for primordial black holes that evolve from being in radiation-dominated universes to being in matter-dominated universes.

5.1 Matching Einstein-de Sitter Universes

The Robertson-Walker metric for the Einstein-de Sitter universe is

$$\mathrm{d}s^2 = -\mathrm{d}{t_c}^2 + [R(t_c)]^2(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)), \tag{5.1}$$

where the solution of the Friedmann equations yields that $R(t_c)$ will be given by $(2H_o t_c)^{1/2}$ when the universe starts off radiation dominated, and by $(3H_o(t_c+C)/2)^{2/3}$ when it later becomes matter dominated (the constant of integration C being generally non-zero in the matter-dominated phase to account for the difference in the time it would take the universe to reach a given scale in the radiation-dominated phase from the time it would have taken to reach that scale in the matter-dominated phase).

To show that the spacetimes for the different scale factors can be matched, the Lichnerowicz conditions will be used. The Lichnerowicz conditions require that both the metric g_{ab} and its derivatives $g_{ab|c}$ be continuous across the junction. Equivalently, since the derivatives of the metric are given by

$$g_{ab|c} = g_{da}\Gamma^{d}_{\ bc} + g_{db}\Gamma^{d}_{\ ac}, \tag{5.2}$$

then if the metric is continuous, the derivatives of the metric will be continuous if the metric connections are continuous.

The metric will be continuous if the scale factor is continuous, which requires

$$(2H_o t_c)^{1/2} = \left(\frac{3}{2}H_o(t_c + C)\right)^{2/3} \tag{5.3}$$

at the junction. For any hypersurface of constant time t_c this matching can be achieved with a suitable constant C.

For the radiation-dominated case, the non-zero metric connections are given by

$$\begin{gathered}
\Gamma^{0}_{\ 11} = 1 \\
\Gamma^{0}_{\ 22} = r^2 \\
\Gamma^{0}_{\ 33} = r^2 \sin^2\theta \\
\Gamma^{1}_{\ 01} = \Gamma^{1}_{\ 10} = \Gamma^{2}_{\ 02} = \Gamma^{2}_{\ 20} = \Gamma^{3}_{\ 03} = \Gamma^{3}_{\ 30} = \frac{1}{2H_o t_c} \\
\Gamma^{1}_{\ 22} = -r \\
\Gamma^{1}_{\ 33} = -r\sin^2\theta \\
\Gamma^{2}_{\ 12} = \Gamma^{2}_{\ 21} = \Gamma^{3}_{\ 13} = \Gamma^{3}_{\ 31} = \frac{1}{r} \\
\Gamma^{2}_{\ 33} = -\cos\theta\sin\theta \\
\Gamma^{3}_{\ 23} = \Gamma^{3}_{\ 32} = \cot\theta,
\end{gathered} \tag{5.4}$$

and for the matter-dominated case, the non-zero metric connections are given by

$$\Gamma^{0}_{\ 11} = \left(\frac{3}{2}H_o(t_c + C)\right)^{1/3}$$

$$\Gamma^{0}_{\ 22} = r^2\left(\frac{3}{2}H_o(t_c + C)\right)^{1/3}$$

$$\Gamma^0_{33} = r^2 \sin^2\theta \left(\frac{3}{2}H_o(t_c + C)\right)^{1/3}$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{2}{3H_o(t_c + C)}$$

$$\Gamma^1_{22} = -r$$

$$\Gamma^1_{33} = -r\sin^2\theta$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^2_{33} = -\cos\theta\sin\theta$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta. \tag{5.5}$$

Thus, the derivatives of the metric will match across the boundary if the conditions

$$1 = \left(\frac{3}{2}H_o(t_c + C)\right)^{1/3} \tag{5.6}$$

and

$$\frac{1}{2H_o t_c} = \frac{2}{3H_o(t_c + C)} \tag{5.7}$$

are satisfied simultaneously with the condition that the metric be continuous. It can be seen that the Lichnerowicz conditions will be satisfied if $H_o t_c = 1/2$ and $H_o C = 1/6$. However, it is already known that $H_o t_o = 1/2$ for a radiation-dominated universe for any choice of time t_o at which H_o is evaluated and the scale factor is set equal to 1. So the Lichnerowicz conditions only require that $R = 1$ at the junction, which can arbitrarily be specified to be true at any time, which means the matching may in fact occur at any time. In theory it should be allowed for energy to be converted between radiation and matter, so it makes sense that this instantaneous matching works. In Figure 5.1 it can be seen that both the scale factor and its slope match at $H_o t_c = 1/2$ with $H_o C = 1/6$, allowing the radiation-dominated universe to smoothly match onto the matter-dominated universe. It should also be apparent that one could do the converse and match an initially matter-dominated universe onto a radiation-dominated universe at $H_o t_c = 2/3$ with $H_o C = -1/6$, which could be used to realistically model the situation of a universe initially filled with matter and antimatter that annihilates to leave radiation.

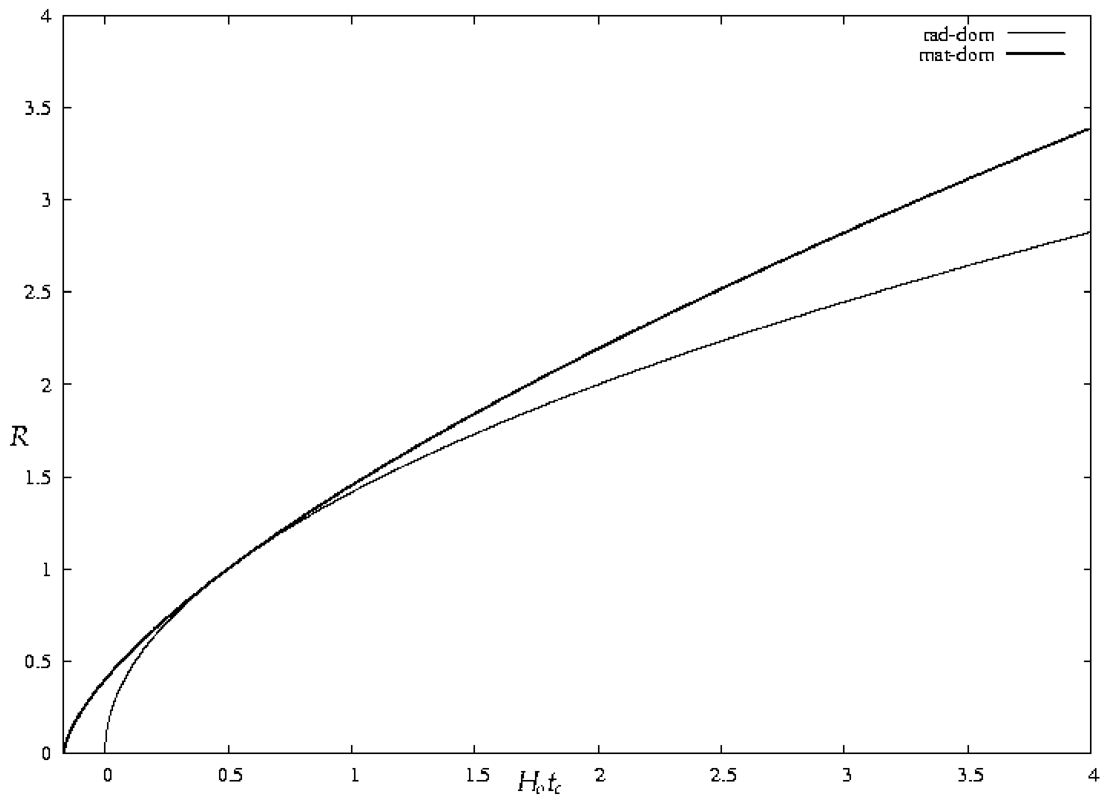

Figure 5.1: Evolution of the scale factor versus time, with the thin line representing the curve for a radiation-dominated universe, $R = (2H_o t_c)^{1/2}$, and the thick line representing the curve for a matter-dominated universe, $R = (3H_o(t_c + 1/6)/2)^{2/3}$. The scale factor and its slope both match when $R = 1$ at $H_o t_c = 1/2$.

The Robertson-Walker metric for the Einstein-de Sitter universe in conformal form is

$$\mathrm{d}s^2 = [R(t)]^2(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)), \tag{5.8}$$

where $R(t)\mathrm{d}t = \mathrm{d}t_c$. The scale factor will be given by $H_o t$ when the universe starts off radiation dominated, and by $(H_o(t+C)/2)^2$ when it later becomes matter dominated. It can be shown that the Lichnerowicz conditions for matching across a constant t hypersurface that correspond to $H_o t_c = 1/2$ and $H_o C = 1/6$ are that $H_o t = H_o C = 1$.

5.2 Matching Kerr-Schild Cosmological Black Hole Backgrounds

The metric for an asymptotically-Einstein-de Sitter Reissner-Nordström black hole that expands with the universe is given by

$$ds^2 = [R(t)]^2 \left(-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) + \left(\frac{2m}{r} - \frac{e^2}{r^2} \right) (dt + dr)^2 \right), \quad (5.9)$$

where $R(t)$ will be given by $H_o t$ when the universe starts off radiation dominated, and by $(H_o(t+C)/2)^2$ when it later becomes matter dominated.

The condition for the metric to match across the junction is that

$$H_o t = \left(\frac{1}{2} H_o (t + C) \right)^2 . \quad (5.10)$$

The non-zero metric connections are given by

$$\Gamma^0_{00} = \frac{(e^4 - 4e^2mr + 4m^2r^2 + r^4)\dot{R}r + (e^2 - mr)(e^2 - 2mr)R}{Rr^5}$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{((e^2 - mr)R + (e^2 - 2mr)\dot{R}r)(e^2 - 2mr - r^2)}{Rr^5}$$

$$\Gamma^0_{11} = \frac{(e^2 - mr)(e^2 - 2mr - 2r^2)R + (e^2 - 2mr - r^2)^2\dot{R}r}{Rr^5}$$

$$\Gamma^0_{22} = -\frac{(e^2 - 2mr - r^2)\dot{R}r - (e^2 - 2mr)R}{Rr}$$

$$\Gamma^0_{33} = -\sin^2\theta \, \frac{(e^2 - 2mr - r^2)\dot{R}r - (e^2 - 2mr)R}{Rr}$$

$$\Gamma^1_{00} = -\frac{((e^2 - mr)R + (e^2 - 2mr)\dot{R}r)(e^2 - 2mr + r^2)}{Rr^5}$$

$$\Gamma^1_{01} = \Gamma^1_{10} = -\frac{(e^2 - mr)(e^2 - 2mr)R + (e^2 - 2mr + r^2)(e^2 - 2mr - r^2)\dot{R}r)}{Rr^5}$$

$$\Gamma^1_{11} = -\frac{((e^2 - mr)R + (e^2 - 2mr)\dot{R}r)(e^2 - 2mr - r^2)}{Rr^5}$$

$$\Gamma^1_{22} = -\frac{(e^2 - 2mr + r^2)R - (e^2 - 2mr)\dot{R}r}{Rr}$$

$$\Gamma^1_{33} = -\sin^2\theta \, \frac{(e^2 - 2mr + r^2)R - (e^2 - 2mr)\dot{R}r}{Rr}$$

$$
\begin{aligned}
\Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} &= \frac{\dot{R}}{R} \\
\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} &= \frac{1}{r} \\
\Gamma^2_{33} &= -\cos\theta\sin\theta \\
\Gamma^3_{23} = \Gamma^3_{32} &= \cot\theta,
\end{aligned}
\tag{5.11}
$$

so to satisfy the Lichnerowicz conditions, the only additional condition is that $\dot{R}$ match at the junction. This requires that

$$
H_o = \left(\frac{1}{2}H_o\right)^2 2(t+C), \tag{5.12}
$$

which together with the condition that R match will be satisfied for $H_o t = H_o C = 1$, and since $H_o t_o = 1$ for any time t_o in the radiation-dominated era that H_o is measured and R is set equal to 1, then the matching may take place at any time.

The metric for an asymptotically Einstein-de Sitter Reissner-Nordström black hole that doesn't expand with the universe is given by

$$
\mathrm{d}s^2 = [R(t)]^2\left(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right) + \left(\frac{2m}{r} - \frac{e^2}{r^2}\right)(\mathrm{d}t + \mathrm{d}r)^2. \tag{5.13}
$$

Once again, the metric will match across the junction with

$$
H_o t = \left(\frac{1}{2}H_o(t+C)\right)^2. \tag{5.14}
$$

The non-zero metric connections are given by

$$
\begin{aligned}
\Gamma^0_{00} &= -\frac{\left((e^2-2mr) - R^2r^2\right)\dot{R}Rr^3 - e^4 + 3e^2mr - 2m^2r^2}{R^4r^5} \\
\Gamma^0_{01} = \Gamma^0_{10} &= -\frac{(e^2-mr)R^2r^2 + (e^2-2mr)\dot{R}Rr^3 - e^4 + 3e^2mr - 2m^2r^2}{R^4r^5} \\
\Gamma^0_{11} &= -\frac{\left((e^2-2mr) - R^2r^2\right)\dot{R}Rr^3 + 2(e^2-mr)R^2r^2 - e^4 + 3e^2mr - 2m^2r^2}{R^4r^5} \\
\Gamma^0_{22} &= \frac{(R^2r^2 - e^2 + 2mr)\dot{R}r + (e^2-2mr)R}{R^3r} \\
\Gamma^0_{33} &= \sin^2\theta\,\frac{(R^2r^2 - e^2 + 2mr)\dot{R}r + (e^2-2mr)R}{R^3r}
\end{aligned}
$$

$$\Gamma^1_{00} = -\frac{(e^2 - mr)R^2r^2 - (e^2 - 2mr)\dot{R}Rr^3 + e^4 - 3e^2mr + 2m^2r^2}{R^4r^5}$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \frac{\left((e^2 - 2mr) + R^2r^2\right)\dot{R}Rr^3 - e^4 + 3e^2mr - 2m^2r^2}{R^4r^5}$$

$$\Gamma^1_{11} = \frac{(e^2 - mr)R^2r^2 + (e^2 - 2mr)\dot{R}Rr^3 - e^4 + 3e^2mr - 2m^2r^2}{R^4r^5}$$

$$\Gamma^1_{22} = \frac{(\dot{R}r - R)(e^2 - 2mr) - R^3r^2}{R^3r}$$

$$\Gamma^1_{33} = -\sin^2\theta\,\frac{\left((e^2 - 2mr) + R^2r^2\right)R - (e^2 - 2mr)\dot{R}r}{R^3r}$$

$$\Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{R}}{R}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^2_{33} = -\cos\theta\sin\theta$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta, \tag{5.15}$$

so the only additional constraint needed to satisfy the Lichnerowicz conditions is that $\dot{R}$ match at the junction. As with the expanding black holes, $\dot{R}$ will match with

$$H_o = \left(\frac{1}{2}H_o\right)^2 2(t + C), \tag{5.16}$$

which together with the condition that R match, will occur for $H_ot = H_oC = 1$. As before, this can arbitrarily be satisfied at any time.

5.3 Matching Isotropic Cosmological Black Hole Backgrounds

The metric for the case of an expanding isotropic Reissner-Nordström black hole in an asymptotically Einstein-de Sitter universe is given by

$$\mathrm{d}s^2 = -\left(\frac{1-\frac{m^2}{4r^2}+\frac{e^2}{4r^2}}{\left(1+\frac{m}{2r}\right)^2-\frac{e^2}{4r^2}}\right)^2 \mathrm{d}{t_c}^2 +$$
$$[R(t_c)]^2\left(\left(1+\frac{m}{2r}\right)^2-\frac{e^2}{4r^2}\right)^2\left(\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)\right). \tag{5.17}$$

The non-zero metric connections are given by

$$\Gamma^0_{01}=\Gamma^0_{10}=\frac{4(e^2m+4e^2r-m^3-4m^2r-4mr^2)}{(e^2-m^2+4r^2)(e+m+2r)(e-m-2r)}$$
$$\Gamma^0_{11}=\frac{(e+m+2r)^4(e-m-2r)^4\dot{R}R}{16(e^2-m^2+4r^2)^2r^4}$$
$$\Gamma^0_{22}=\frac{(e+m+2r)^4(e-m-2r)^4\dot{R}R}{16(e^2-m^2+4r^2)^2r^2}$$
$$\Gamma^0_{33}=\sin^2\theta\;\frac{(e+m+2r)^4(e-m-2r)^4\dot{R}R}{16(e^2-m^2+4r^2)^2r^2}$$
$$\Gamma^1_{00}=\frac{64(e^2m+4e^2r-m^3-4m^2r-4mr^2)(e^2-m^2+4r^2)r^4}{(e+m+2r)^5(e-m-2r)^5R^2}$$
$$\Gamma^1_{01}=\Gamma^1_{10}=\Gamma^2_{02}=\Gamma^2_{20}=\Gamma^3_{03}=\Gamma^3_{30}=\frac{\dot{R}}{R}$$
$$\Gamma^1_{11}=-\frac{2(e^2-m^2-2mr)}{(e+m+2r)(e-m-2r)r}$$
$$\Gamma^1_{22}=\frac{(e^2-m^2+4r^2)r}{(e+m+2r)(e-m-2r)}$$
$$\Gamma^1_{33}=\sin^2\theta\;\frac{(e^2-m^2+4r^2)r}{(e+m+2r)(e-m-2r)}$$
$$\Gamma^2_{12}=\Gamma^2_{21}=\Gamma^3_{13}=\Gamma^3_{31}=-\frac{(e^2-m^2+4r^2)}{(e+m+2r)(e-m-2r)r}$$
$$\Gamma^2_{33}=-\cos\theta\sin\theta$$
$$\Gamma^3_{23}=\Gamma^3_{32}=\cot\theta. \tag{5.18}$$

For the Lichnerowicz conditions to be satisfied, then R must match and $\dot{R}$ must match at the junction, which requires that

$$(2H_ot_c)^{1/2}=\left(\frac{3}{2}H_o(t_c+C)\right)^{2/3} \tag{5.19}$$

and

$$(2H_o)^{1/2}\frac{t_c{}^{-1/2}}{2} = \left(\frac{3}{2}H_o\right)^{2/3}\frac{2(t_c+C)^{-1/3}}{3}. \tag{5.20}$$

These conditions will be satisfied for $H_o t_c = 1/2$ and $H_o C = 1/6$, just as for the standard Einstein-de Sitter universe.

M$^{\text{C}}$Vittie's metric for a non-expanding isotropic black hole in an asymptotically Einstein-de Sitter universe (generalized by Gao & Zhang 2004 to the charged case) is given by

$$\mathrm{d}s^2 = -\left(\frac{1-\frac{m^2/R^2}{4r^2}+\frac{e^2/R^2}{4r^2}}{\left(1+\frac{m/R}{2r}\right)^2-\frac{e^2/R^2}{4r^2}}\right)^2 \mathrm{d}t_c{}^2 + \\ [R(t_c)]^2\left(\left(1+\frac{m/R}{2r}\right)^2-\frac{e^2/R^2}{4r^2}\right)^2(\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\;\mathrm{d}\phi^2)). \tag{5.21}$$

The non-zero metric connections are

$$\Gamma^0_{00} = \frac{4(4R^2mr^2-4Re^2r+4Rm^2r-e^2m+m^3)\dot{R}r}{(4R^2r^2+e^2-m^2)(2Rr+e+m)(2Rr-e+m)}$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{4(4R^2mr^2-4Re^2r+4Rm^2r-e^2m+m^3)R}{(4R^2r^2+e^2-m^2)(2Rr+e+m)(2Rr-e+m)}$$

$$\Gamma^0_{11} = \frac{(2Rr+e+m)^3(2Rr-e+m)^3\dot{R}}{16(4R^2r^2+e^2-m^2)R^3r^4}$$

$$\Gamma^0_{22} = \frac{(2Rr+e+m)^3(2Rr-e+m)^3\dot{R}}{16(4R^2r^2+e^2-m^2)R^3r^2}$$

$$\Gamma^0_{33} = \sin^2\theta\;\frac{(2Rr+e+m)^3(2Rr-e+m)^3\dot{R}}{16(4R^2r^2+e^2-m^2)R^3r^2}$$

$$\Gamma^1_{00} = \frac{64(4R^2mr^2-4Re^2r+4Rm^2r-e^2m+m^3)(4R^2r^2+e^2-m^2)R^3r^4}{(2Rr+e+m)^5(2Rr-e+m)^5}$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{(4R^2r^2+e^2-m^2)\dot{R}}{(2Rr+e+m)(2Rr-e+m)R}$$

$$\Gamma^1_{11} = -\frac{2(2Rmr-e^2+m^2)}{(2Rr+e+m)(2Rr-e+m)r}$$

$$\Gamma^1_{22} = -\frac{(4R^2r^2+e^2-m^2)r}{(2Rr+e+m)(2Rr-e+m)}$$

$$
\begin{aligned}
\Gamma^1_{33} &= -\sin^2\theta\,\frac{(4R^2r^2+e^2-m^2)r}{(2Rr+e+m)(2Rr-e+m)}\\
\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} &= \frac{4R^2r^2+e^2-m^2}{(2Rr+e+m)(2Rr-e+m)r}\\
\Gamma^2_{33} &= -\cos\theta\sin\theta\\
\Gamma^3_{23} = \Gamma^3_{32} &= \cot\theta. \qquad (5.22)
\end{aligned}
$$

Thus, the Lichnerowicz conditions for matching the radiation-dominated and matter-dominated cases simply require that R match and $\dot{R}$ match at the boundary, so

$$(2H_o t_c)^{1/2} = \left(\frac{3}{2}H_o(t_c+C)\right)^{2/3} \qquad (5.23)$$

and

$$(2H_o)^{1/2}\frac{{t_c}^{-1/2}}{2} = \left(\frac{3}{2}H_o\right)^{2/3}\frac{2(t_c+C)^{-1/3}}{3}, \qquad (5.24)$$

which will once again be satisfied for $H_o t_c = 1/2$ and $H_o C = 1/6$, just as for the standard Einstein-de Sitter universe. Looking at the metrics for the cosmological black hole spacetimes, the time derivatives should just yield $\dot{R}$ in the metric connections as with the standard Einstein-de Sitter universe, so it makes sense that the cosmological black holes can be matched just as easily as the standard Einstein-de Sitter universe can.

Chapter 6

The Influence of Cosmological Black Holes on the Universe's Expansion

In this chapter the Weyl curvature will be calculated for the cosmological black holes, since Weyl curvature should lead to shear that could influence the volume expansion of the universe. The volume expansion will then be calculated for co-ordinate volumes and independently by calculating the velocity field of the matter (since the matter may be flowing rather than tied to the co-ordinates), and the independent calculations will be compared to ensure they are consistent with each other. The calculation of the velocity field will enable the shear and acceleration to be determined to see how they influence the volume expansion.

6.1 Weyl Curvature

Since Weyl curvature is the relativistic equivalent of tidal force, then the Weyl curvature in the cosmological black hole spacetimes should be expected to introduce shear in the velocity field. Thus, the Weyl curvature of the cosmological black holes will be calculated in this section.

The Weyl scalar for the expanding Kerr-Schild cosmological black holes (Equations 3.26 and 3.53) is

$$C^{abcd}C_{abcd} = \frac{48(e^2 - mr)^2}{R^4 r^8}, \tag{6.1}$$

so it is like the Weyl scalar for an isolated black hole

$$C^{abcd}C_{abcd} = \frac{48(e^2 - mr)^2}{r^8}, \tag{6.2}$$

but scaled down by R^4, so that not only does the Weyl curvature fall off from infinity at $r = 0$ to zero at $r = \infty$ (or if $e^2 = mr$), it falls off from infinity at $t = 0$ to zero at $t = \infty$.

The Weyl scalar for the non-expanding Kerr-Schild cosmological black holes (Equations 3.33 and 3.57) is

$$C^{abcd}C_{abcd} = \frac{16}{3}\frac{\left((\ddot{R}R - 5\dot{R}^2)(e^2r^2 - 2mr^3) + 2\dot{R}R(3e^2r - 4mr^2) - 3R^2(e^2 - mr)\right)^2}{R^{20}r^8}, \tag{6.3}$$

which in the case of a radiation-dominated universe is

$$C^{abcd}C_{abcd} = \frac{16}{3}\frac{((10r^2 - 8rR + 3R^2)mr - (5r^2 - 6rR + 3R^2)e^2)^2}{r^8 R^{20}} \tag{6.4}$$

and in the case of a matter-dominated universe is

$$C^{abcd}C_{abcd} = \frac{16}{3}\frac{\left((36r^2 - 16rR^{1/2} + 3R)mr - 3(6r^2 - 4rR^{1/2} + R)e^2\right)^2}{r^8 R^{18}}, \tag{6.5}$$

so the influence of the scale factor causes the Weyl curvature to approach zero even more quickly for the non-expanding black holes.

The Weyl scalar for the constant-mass isotropic cosmological black holes (Equation 4.28) is

$$C^{abcd}C_{abcd} = \frac{196608\left((m + 4r)e^2 - m(m + 2r)^2\right)^2 r^6}{(e + m + 2r)^8(e - m - 2r)^8 R^4}, \tag{6.6}$$

which simplifies to

$$C^{abcd}C_{abcd} = \frac{196608(m + 2r)^4 r^6}{(m + 2r)^{16} R^4} = \frac{48m^2}{r^6 R^4 (1 + \frac{m}{2r})^{12}} \tag{6.7}$$

in the uncharged case. While this resembles the Weyl curvature for a Schwarzschild black hole, the presence of the $(1 + m/(2r))^{12}$ term in the denominator means that as

r approaches zero, the denominator approaches infinity so that the Weyl curvature is zero at the origin instead of infinite, which makes sense because the region inside the event horizon in the isotropic black hole spacetime is a remapping of the region outside the event horizon in the standard Schwarzschild spacetime so there should be zero Weyl curvature at the origin. While the pressure (in the radiation-dominated case) and energy density become infinite at the event horizon, the event horizon doesn't take on the role of a curvature singularity like the singularity at the origin of an isolated Schwarzschild black hole. The Weyl curvature is finite at the event horizon and is infinite only at the Big Bang.

The Weyl scalar for the M$^{\text{c}}$Vittie, or more generally the Gao & Zhang black holes (Equation 4.36) is

$$C^{abcd}C_{abcd} = \frac{196608\left((m+4rR)e^2 - m(m+2r)^2\right)^2 r^6 R^6}{(e+m+2rR)^8(e-m-2rR)^8}, \tag{6.8}$$

which simplifies to

$$C^{abcd}C_{abcd} = \frac{196608 m^2 (m+2rR)^4 r^6 R^6}{(m+2rR)^{16}} = \frac{48m^2}{r^6 R^6 \left(1+\frac{m}{2rR}\right)^{12}} \tag{6.9}$$

in the uncharged case, analogous to the constant-mass isotropic black holes (cf. Equation 6.7). As with the case of the constant-mass black holes, the Weyl curvature is zero at the origin, but unlike the constant-mass black holes, the Weyl curvature is zero at the Big Bang.

6.2 Volume Expansion

The volume in some comoving region Ω is given by

$$V = \int_\Omega \sqrt{\gamma}\, \mathrm{d}^3x, \tag{6.10}$$

where γ is the determinant of the spatial part of the metric tensor.

The volume integrated over a given radius range r_a to r_b for the expanding Kerr-Schild black holes (Equations 3.26 and 3.53) will be

$$V = \int_{r_a}^{r_b}\int_0^{\pi}\int_0^{2\pi} \sqrt{R^6\left(1+\frac{2m}{r}-\frac{e^2}{r^2}\right) r^4 \sin^2\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi}, \tag{6.11}$$

which simplifies to

$$V = 4\pi R^3 \int_{r_a}^{r_b} \sqrt{1 + \frac{2m}{r} - \frac{e^2}{r^2}}\, r^2 \,\mathrm{d}r. \tag{6.12}$$

Taking the derivative of the volume with respect to the co-ordinate time yields

$$\frac{\dot{V}}{V} = \frac{4\pi 3 R^2 \dot{R} \int_{r_a}^{r_b} \sqrt{1 + \frac{2m}{r} - \frac{e^2}{r^2}}\, r^2 \,\mathrm{d}r}{4\pi R^3 \int_{r_a}^{r_b} \sqrt{1 + \frac{2m}{r} - \frac{e^2}{r^2}}\, r^2 \,\mathrm{d}r} = 3\frac{\dot{R}}{R}. \tag{6.13}$$

Thus, for the expanding Kerr-Schild black holes, the volume goes as the cube of the scale factor, so the volume expansion is unaffected by the black hole and is the same as it is for the plain FRW universe.

For the non-expanding Kerr-Schild cosmological black holes (Equations 3.33 and 3.57), the volume will be

$$V = \int_{r_a}^{r_b} \int_0^{\pi} \int_0^{2\pi} \sqrt{R^6 \left(1 + \frac{1}{R^2}\left(\frac{2m}{r} - \frac{e^2}{r^2}\right)\right)}\, r^4 \sin^2\theta \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\phi, \tag{6.14}$$

which simplifies to

$$V = 4\pi R^3 \int_{r_a}^{r_b} \sqrt{1 + \frac{1}{R^2}\left(\frac{2m}{r} - \frac{e^2}{r^2}\right)}\, r^2 \,\mathrm{d}r. \tag{6.15}$$

Taking the time derivative in this case yields

$$\frac{\dot{V}}{V} = \frac{4\pi 3 R^2 \dot{R} \int_{r_a}^{r_b} \sqrt{1 + \frac{1}{R^2}\left(\frac{2m}{r} - \frac{e^2}{r^2}\right)}\, r^2 \,\mathrm{d}r + 4\pi R^3 \int_{r_a}^{r_b} \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{1 + \frac{1}{R^2}\left(\frac{2m}{r} - \frac{e^2}{r^2}\right)}\right) r^2 \,\mathrm{d}r}{4\pi R^3 \int_{r_a}^{r_b} \sqrt{1 + \frac{1}{R^2}\left(\frac{2m}{r} - \frac{e^2}{r^2}\right)}\, r^2 \,\mathrm{d}r}$$

$$= 3\frac{\dot{R}}{R} - \frac{\dot{R}}{R^3} \frac{\int_{r_a}^{r_b} \frac{\frac{2m}{r} - \frac{e^2}{r^2}}{\sqrt{1 + \frac{1}{R^2}(\frac{2m}{r} - \frac{e^2}{r^2})}}\, r^2 \,\mathrm{d}r}{\int_{r_a}^{r_b} \sqrt{1 + \frac{1}{R^2}\left(\frac{2m}{r} - \frac{e^2}{r^2}\right)}\, r^2 \,\mathrm{d}r}. \tag{6.16}$$

Thus, the volume expansion appears to be smaller than what is expected for an FRW universe, suggesting the volume may be sheared by the black hole.

In the case where the charge is set to zero

$$\frac{\dot{V}}{V} = 3\frac{\dot{R}}{R} - \frac{2m\dot{R}}{R^3} \frac{\int_{r_a}^{r_b} \left(\sqrt{1 + \frac{2m}{R^2 r}}\right)^{-1} r \,\mathrm{d}r}{\int_{r_a}^{r_b} \sqrt{1 + \frac{2m}{R^2 r}}\, r^2 \,\mathrm{d}r}, \tag{6.17}$$

which integrates to

$$\frac{\dot{V}}{V} = 3\frac{\dot{R}}{R} - \frac{2m\dot{R}}{R^3}\frac{\frac{1}{8}\left[\left(4r - 6\frac{2m}{R^2}\right)\sqrt{r^2 + \frac{2mr}{R^2}} + 3\left(\frac{2m}{R^2}\right)^2 \log\frac{\sqrt{r+\frac{2m}{R^2}}+\sqrt{r}}{\sqrt{r+\frac{2m}{R^2}}-\sqrt{r}}\right]_{r_a}^{r_b}}{\frac{1}{48}\left[\left(16r^2 + 4r\frac{2m}{R^2} - 6\left(\frac{2m}{R^2}\right)^2\right)\sqrt{r^2 + \frac{2mr}{R^2}} + 3\left(\frac{2m}{R^2}\right)^3 \log\frac{\sqrt{r+\frac{2m}{R^2}}+\sqrt{r}}{\sqrt{r+\frac{2m}{R^2}}-\sqrt{r}}\right]_{r_a}^{r_b}}. \tag{6.18}$$

Evaluating from $r_a = 0$ to $r_b = r$ yields

$$\frac{\dot{V}}{V} = 3\frac{\dot{R}}{R} - 6\frac{\dot{R}}{R}\frac{\left(4r\frac{2m}{R^2} - 6\left(\frac{2m}{R^2}\right)^2\right)\sqrt{r^2 + \frac{2mr}{R^2}} + 3\left(\frac{2m}{R^2}\right)^3 \log\frac{\sqrt{r+\frac{2m}{R^2}}+\sqrt{r}}{\sqrt{r+\frac{2m}{R^2}}-\sqrt{r}}}{\left(16r^2 + 4r\frac{2m}{R^2} - 6\left(\frac{2m}{R^2}\right)^2\right)\sqrt{r^2 + \frac{2mr}{R^2}} + 3\left(\frac{2m}{R^2}\right)^3 \log\frac{\sqrt{r+\frac{2m}{R^2}}+\sqrt{r}}{\sqrt{r+\frac{2m}{R^2}}-\sqrt{r}}}, \tag{6.19}$$

which is found to vary monotonically from $\dot{V}/V = 2\dot{R}/R$ to $\dot{V}/V = 3\dot{R}/R$ as r goes from zero to infinity. Thus as r approaches zero, the volume goes as the square of the scale factor, unlike normal FRW, but as r approaches infinity the volume goes as the cube of the scale factor, approaching the usual FRW volume expansion. In the vicinity of the black hole the volume expansion will be decreased, but the volume expansion of the universe as a whole will not be affected.

Considering a thin shell of infinitesimal thickness Δr, then based on Equation 6.16 the volume expansion as a function of radius is given by

$$\frac{\dot{V}}{V} = 3\frac{\dot{R}}{R} - \frac{\dot{R}}{R^3}\frac{\frac{\frac{2m}{r} - \frac{e^2}{r^2}}{\sqrt{1+\frac{1}{R^2}\left(\frac{2m}{r} - \frac{e^2}{r^2}\right)}}\, r^2\, \Delta r}{\sqrt{1 + \frac{1}{R^2}\left(\frac{2m}{r} - \frac{e^2}{r^2}\right)}\, r^2\, \Delta r} = 3\frac{\dot{R}}{R} - \frac{\dot{R}}{R^3}\frac{\frac{2m}{r} - \frac{e^2}{r^2}}{1 + \frac{1}{R^2}\left(\frac{2m}{r} - \frac{e^2}{r^2}\right)}, \tag{6.20}$$

which neglecting charge (and with m and R both finite), yields $\dot{V}/V = 2\dot{R}/R$ for small r or

$$\frac{\dot{V}}{V} = 3\frac{\dot{R}}{R}\left(1 - \frac{2m}{3rR^2}\right) \tag{6.21}$$

for large r, so that $\dot{V}/V = 3\dot{R}/R$ as r goes to infinity. This is consistent with the volume expansion found for a sphere of radius r above.

For the constant-mass isotropic cosmological black holes (Equation 4.28) the volume will be

$$V = \int_{r_a}^{r_b}\int_0^{\pi}\int_0^{2\pi}\sqrt{R^6\left(\left(1+\frac{m}{2r}\right)^2-\frac{e^2}{4r^2}\right)^6 r^4\sin^2\theta \, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi}, \tag{6.22}$$

which simplifies to

$$V = 4\pi R^3\int_{r_a}^{r_b}\left(\left(1+\frac{m}{2r}\right)^2-\frac{e^2}{4r^2}\right)^3 r^2\, \mathrm{d}r. \tag{6.23}$$

As before with the expanding Kerr-Schild black holes this leads to

$$\frac{\dot{V}}{V} = \frac{4\pi 3R^2\dot{R}\int_{r_a}^{r_b}\left(\left(1+\frac{m}{2r}\right)^2-\frac{e^2}{4r^2}\right)^3 r^2\, \mathrm{d}r}{4\pi R^3\int_{r_a}^{r_b}\left(\left(1+\frac{m}{2r}\right)^2-\frac{e^2}{4r^2}\right)^3 r^2\, \mathrm{d}r} = 3\frac{\dot{R}}{R} \tag{6.24}$$

so that the volume goes as the cube of the scale factor, and the volume expansion is unaffected by the black hole.

For the M$^{\mathrm{c}}$Vittie (Gao & Zhang) black holes (Equation 4.36), the volume will be

$$V = \int_{r_a}^{r_b}\int_0^{\pi}\int_0^{2\pi}\sqrt{R^6\left(\left(1+\frac{m}{2rR}\right)^2-\frac{e^2}{4r^2R^2}\right)^6 r^4\sin^2\theta \, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi}, \tag{6.25}$$

which simplifies to

$$V = 4\pi R^3\int_{r_a}^{r_b}\left(\left(1+\frac{m}{2rR}\right)^2-\frac{e^2}{4r^2R^2}\right)^3 r^2\, \mathrm{d}r. \tag{6.26}$$

Thus,

$$\begin{aligned}\frac{\dot{V}}{V} &= \frac{4\pi 3R^2\dot{R}\int_{r_a}^{r_b}\left(\left(1+\frac{m}{2rR}\right)^2-\frac{e^2}{4r^2R^2}\right)^3 r^2\, \mathrm{d}r}{4\pi R^3\int_{r_a}^{r_b}\left(\left(1+\frac{m}{2rR}\right)^2-\frac{e^2}{4r^2R^2}\right)^3 r^2\, \mathrm{d}r} + \\ &\qquad \frac{4\pi R^3\int_{r_a}^{r_b}\frac{\mathrm{d}}{\mathrm{d}t_c}\left[\left(\left(1+\frac{m}{2rR}\right)^2-\frac{e^2}{4r^2R^2}\right)^3\right] r^2\, \mathrm{d}r}{4\pi R^3\int_{r_a}^{r_b}\left(\left(1+\frac{m}{2rR}\right)^2-\frac{e^2}{4r^2R^2}\right)^3 r^2\, \mathrm{d}r} \\ &= 3\frac{\dot{R}}{R} + \frac{6\dot{R}\int_{r_a}^{r_b}\left(\left(1+\frac{m}{2rR}\right)^2-\frac{e^2}{4r^2R^2}\right)^2\left(-\frac{m}{2rR^2}\left(1+\frac{m}{2rR}\right)+\frac{e^2}{4r^2R^3}\right) r^2\, \mathrm{d}r}{\int_{r_a}^{r_b}\left(\left(1+\frac{m}{2rR}\right)^2-\frac{e^2}{4r^2R^2}\right)^3 r^2\, \mathrm{d}r}.\end{aligned} \tag{6.27}$$

For the uncharged case this is simply

$$\frac{\dot{V}}{V} = 3\frac{\dot{R}}{R} - \frac{3m\dot{R}}{R^2}\frac{\int_{r_a}^{r_b}\left(1+\frac{m}{2rR}\right)^5\ r\ \mathrm{d}r}{\int_{r_a}^{r_b}\left(1+\frac{m}{2rR}\right)^6\ r^2\ \mathrm{d}r}, \tag{6.28}$$

which integrates to yield

$$\frac{\dot{V}}{V} = 3\frac{\dot{R}}{R}\left(1 - \frac{m}{R}\frac{[A]_{r_a}^{r_b}}{[B]_{r_a}^{r_b}}\right), \tag{6.29}$$

where A and B are given by

$$A = \frac{240\log(r)R^3m^2r^3 + 48R^5r^5 + 240R^4mr^4 - 120R^2m^3r^2 - 15Rm^4r - m^5}{96R^5r^3} \tag{6.30}$$

and

$$B = \frac{480\log(r)R^3m^3r^3 + 64R^6r^6 + 288R^5mr^5 + 720R^4m^2r^4}{192R^6r^3} - \frac{180R^2m^4r^2 + 18Rm^5r + m^6}{192R^6r^3}. \tag{6.31}$$

The volume expansion can't be evaluated starting from $r_a = 0$, so evaluating from $r_a = m/(2R)$ to $r_b = r$ yields

$$\frac{\dot{V}}{V} = 3\frac{\dot{R}}{R}\left(1 - \frac{m}{R}\frac{A - \left(\frac{5}{2}\log\left(\frac{m}{2R}\right)\left(\frac{m}{R}\right)^2 - \frac{11}{6}\left(\frac{m}{R}\right)^2\right)}{B - \frac{5}{2}\log\left(\frac{m}{2R}\right)\left(\frac{m}{R}\right)^3}\right). \tag{6.32}$$

For large values of r (with m and R both finite), this goes as

$$\frac{\dot{V}}{V} = 3\frac{\dot{R}}{R}\left(1 - \frac{3m}{2rR}\right), \tag{6.33}$$

which in the limit as r goes to infinity approaches $\dot{V}/V = 3\dot{R}/R$, so the volume expansion only asymptotically approaches that of FRW as r goes to infinity.

Considering a thin shell of infinitesimal thickness Δr, then based on Equation 6.27 the volume expansion as a function of radius is given by

$$\begin{aligned}\frac{\dot{V}}{V} &= 3\frac{\dot{R}}{R} + \frac{6\dot{R}\left(\left(1+\frac{m}{2rR}\right)^2 - \frac{e^2}{4r^2R^2}\right)^2\left(-\frac{m}{2rR^2}\left(1+\frac{m}{2rR}\right) + \frac{e^2}{4r^2R^3}\right)\ r^2\ \Delta r}{\left(\left(1+\frac{m}{2rR}\right)^2 - \frac{e^2}{4r^2R^2}\right)^3\ r^2\ \Delta r} \\ &= 3\frac{\dot{R}}{R} + \frac{6\dot{R}\left(-\frac{m}{2rR^2}\left(1+\frac{m}{2rR}\right) + \frac{e^2}{4r^2R^3}\right)}{\left(\left(1+\frac{m}{2rR}\right)^2 - \frac{e^2}{4r^2R^2}\right)},\end{aligned} \tag{6.34}$$

which neglecting charge yields $\dot{V}/V = -3\dot{R}/R$ in the limit of small r and

$$\frac{\dot{V}}{V} = 3\frac{\dot{R}}{R}\left(1 - \frac{m}{rR}\right) \tag{6.35}$$

for large r, which as r goes to infinity approaches $\dot{V}/V = 3\dot{R}/R$. The decrease in the volume expansion of the shell is slightly less than that of the essentially spherical volume of radius r (Equation 6.33), which makes sense if the volume expansion is being affected more near the black hole than at the boundary of the sphere.

While the decrease in the volume expansion compared with FRW suggests there may be shear in the expanding Kerr-Schild and M$^{\text{C}}$Vittie cosmological black holes and not in the non-expanding Kerr-Schild and constant-mass isotropic cosmological black holes, these calculations only show what happens to a volume defined by a constant value of the co-ordinate r and in terms of the co-ordinate time rather than the proper time. If matter is streaming in or out of the volume, then considering a region defined by r doesn't really specify what is happening in terms of what an observer would see the matter doing. Also, considering Raychaudhuri's equation, the volume expansion may be decreased (or increased) relative to FRW by an increase (or decrease) in the mass-energy, rather than just the shear due to the inhomogeneity, and if the flow of the velocity field isn't geodesic, then there may also be acceleration (or deceleration) that may increase (or decrease) the volume expansion. Thus, it is necessary to determine the velocity field of the matter to determine whether shear exists that slows the volume expansion.

6.3 The Velocity Field

The approach used to solve for the velocity field in this section is based on the approach of Sultana (2003), which was used to obtain the velocity field of a cosmological white hole in the case of pressureless dust.

For a perfect fluid plus heat conduction and an electric field, the total energy momentum tensor is

$$T^{ab} = (\mu + p)u^a u^b + pg^{ab} + q^a u^b + u^a q^b + T^{ab}_{(e)}, \tag{6.36}$$

where $T^{ab}_{(e)}$ is the energy-momentum tensor component corresponding to the electric field. Since

$$\left((\mu+p)(u^0)^2+2q^0u^0\right)(u^1)^2+\left((\mu+p)(u^1)^2+2q^1u^1\right)(u^0)^2 \\ = 2\left((\mu+p)u^0u^1+q^0u^1+u^0q^1\right)u^0u^1, \tag{6.37}$$

then

$$(T^{00}-pg^{00}-T^{00}_{(e)})(u^1)^2+(T^{11}-pg^{11}-T^{11}_{(e)})(u^0)^2=2(T^{01}-pg^{01}-T^{01}_{(e)})u^0u^1. \tag{6.38}$$

Since there is no vorticity, the velocity field u^a should only contain a temporal component u^0 and a radial component u^1, and since $u^au_a=-1$,

$$g_{ab}u^au^b=g_{00}(u^0)^2+2g_{01}u^0u^1+g_{11}(u^1)^2=-1. \tag{6.39}$$

Multiplying Equation 6.39 by $(T^{01}-pg^{01}-T^{01}_{(e)})$ gives

$$(T^{01}-pg^{01}-T^{01}_{(e)})g_{00}(u^0)^2+2(T^{01}-pg^{01}-T^{01}_{(e)})g_{01}u^0u^1+ \\ (T^{01}-pg^{01}-T^{01}_{(e)})g_{11}(u^1)^2=-(T^{01}-pg^{01}-T^{01}_{(e)}), \tag{6.40}$$

and multiplying Equation 6.38 by g_{01} gives

$$(T^{00}-pg^{00}-T^{00}_{(e)})(u^1)^2g_{01}+(T^{11}-pg^{11}-T^{11}_{(e)})(u^0)^2g_{01}=2(T^{01}-pg^{01}-T^{01}_{(e)})u^0u^1g_{01}. \tag{6.41}$$

Summing Equations 6.40 and 6.41 yields

$$\left((T^{00}-pg^{00}-T^{00}_{(e)})g_{01}+(T^{01}-pg^{01}-T^{01}_{(e)})g_{11}\right)(u^1)^2+ \\ \left((T^{11}-pg^{11}-T^{11}_{(e)})g_{01}+(T^{01}+pg^{01}-T^{01}_{(e)})g_{00}\right)(u^0)^2=-(T^{01}-pg^{01}-T^{01}_{(e)}), \tag{6.42}$$

so

$$(u^1)^2=\frac{-(T^{01}-pg^{01}-T^{01}_{(e)})-\left((T^{00}-pg^{00}-T^{00}_{(e)})g_{01}+(T^{01}-pg^{01}-T^{01}_{(e)})g_{11}\right)(u^0)^2}{(T^{00}-pg^{00}-T^{00}_{(e)})g_{01}+(T^{01}-pg^{01}-T^{01}_{(e)})g_{11}}. \tag{6.43}$$

Defining α and β as

$$\alpha=-\frac{T^{01}-pg^{01}-T^{01}_{(e)}}{(T^{00}-pg^{00}-T^{00}_{(e)})g_{01}+(T^{01}-pg^{01}-T^{01}_{(e)})g_{11}} \tag{6.44}$$

and

$$\beta = -\frac{\left((T^{00} - pg^{00} - T^{00}_{(e)})g_{01} + (T^{01} - pg^{01} - T^{01}_{(e)})g_{11}\right)}{(T^{00} - pg^{00} - T^{00}_{(e)})g_{01} + (T^{01} - pg^{01} - T^{01}_{(e)})g_{11}}, \tag{6.45}$$

then Equation 6.43 can be written more simply as

$$(u^1)^2 = \alpha + \beta(u^0)^2. \tag{6.46}$$

Substituting Equation 6.46 into Equation 6.39, rearranging, and squaring yields

$$4(g_{01})^2(u^0)^2(\alpha + \beta(u^0)^2) = (-1 - g_{00}(u^0)^2 - g_{11}(\alpha + \beta(u^0)^2))^2. \tag{6.47}$$

Solving Equation 6.47 for $(u^0)^2$ yields

$$(u^0)^2 = \frac{-((1+\alpha g_{11})(\beta g_{11} + g_{00}) - 2\alpha(g_{01})^2)}{(g_{00} + \beta g_{11})^2 - 4(g_{01})^2\beta} \pm$$

$$\frac{\sqrt{((1+\alpha g_{11})(\beta g_{11} + g_{00}) - 2\alpha(g_{01})^2)^2 - (\alpha g_{11} + 1)^2((g_{00} + \beta g_{11})^2 - 4(g_{01})^2\beta)}}{(g_{00} + \beta g_{11})^2 - 4(g_{01})^2\beta}, \tag{6.48}$$

which can then be used to determine $(u^1)^2$ via Equation 6.46.

For the expanding Kerr-Schild Schwarzschild black holes (Equations 3.3 and 3.40) the velocity field in the case of a matter-dominated universe (with $R = (H_o t/2)^2$ defined just as $R = t^2$ for simplicity) is

$$(u^0)^2 = \frac{36(2r-t)m^3 + 6(18r^2 - 3rt - t^2)m^2 + 54r^3 m + 9r^4}{3(4(3r+t)m + 12m^2 + 3r^2)r^2t^4}$$

$$\pm \frac{2\sqrt{3tm}\sqrt{-3(2r-t)m - 4r^2}\,tm}{3(4(3r+t)m + 12m^2 + 3r^2)r^2t^4}$$

$$(u^1)^2 = \frac{36(2r-t)m^3 + 6(6r^2 + 3rt - t^2)m^2 - 12r^2tm}{3(4(3r+t)m + 12m^2 + 3r^2)r^2t^4}$$

$$\mp \frac{2\sqrt{3tm}\sqrt{-3(2r-t)m - 4r^2}\,(6r+t)m}{3(4(3r+t)m + 12m^2 + 3r^2)r^2t^4}, \tag{6.49}$$

and the velocity field in the case of a radiation-dominated universe (with $R = t$ for simplicity) is

$$
\begin{aligned}
(u^0)^2 &= \frac{4(4r-3t)m^3+3(8r^2-2rt-t^2)m^2+12r^3m+2r^4}{2(2(2r+t)m+4m^2+r^2)r^2t^2} \\
&\pm \frac{\sqrt{tm}\sqrt{-3(4r-3t)m-8r^2}tm}{2(2(2r+t)m+4m^2+r^2)r^2t^2} \\
(u^1)^2 &= \frac{4(4r-3t)m^3+(8r^2+6rt-3t^2)m^2-4r^2tm}{2(2(2r+t)m+4m^2+r^2)r^2t^2} \\
&\mp \frac{\sqrt{tm}\sqrt{-3(4r-3t)m-8r^2}(4r+t)m}{2(2(2r+t)m+4m^2+r^2)r^2t^2}.
\end{aligned} \tag{6.50}
$$

For the non-expanding Kerr-Schild cosmological Schwarzschild black holes (Equations 3.16 and 3.47) the velocity field in the case of a matter-dominated universe is

$$
\begin{aligned}
(u^0)^2 &= \frac{6(7r-2t)r^2t^8m+2(30r^2-11rt+t^2)t^4m^2+12(2r-t)m^3+9r^4t^{12}}{3(4(3r-t)t^4m+12m^2+3r^2t^8)r^2t^8} \\
&\pm 2\frac{\sqrt{m}\sqrt{-(8r^2-2rt-t^2)m-12r^3t^4}t^5m}{3(4(3r-t)t^4m+12m^2+3r^2t^8)r^2t^8} \\
(u^1)^2 &= \frac{-12r^3t^8m+2(18r^2-5rt+t^2)t^4m^2+12(2r-t)m^3}{3(4(3r-t)t^4m+12m^2+3r^2t^8)r^2t^8} \\
&\mp \frac{2\sqrt{m}\sqrt{-(8r^2-2rt-t^2)m-12r^3t^4}(6r-t)t^4m^2}{3(4(3r-t)t^4m+12m^2+3r^2t^8)r^2t^8},
\end{aligned} \tag{6.51}
$$

and the velocity field in the case of a radiation-dominated universe is

$$
\begin{aligned}
(u^0)^2 &= \frac{(11r-4t)r^2t^4m+(20r^2-9rt+t^2)t^2m^2+4(3r-t)m^3+2r^4t^6}{(2(2(2r-t)t^2m+4m^2+r^2t^4)r^2t^4} \\
&\pm \frac{\sqrt{m}\sqrt{-(3r^2+2rt-t^2)m-2r^3t^2}t^3m}{(2(2(2r-t)t^2m+4m^2+r^2t^4)r^2t^4} \\
(u^1)^2 &= \frac{-r^3t^4m+(8r^2-5rt+t^2)t^2m^2+4(3r-t)m^3}{2(2(2r-t)t^2m+4m^2+r^2t^4)r^2t^4} \\
&\mp \frac{\sqrt{m}\sqrt{-(3r^2+2rt-t^2)m-2r^3t^2}(4r-t)t^2m}{2(2(2r-t)t^2m+4m^2+r^2t^4)r^2t^4}.
\end{aligned} \tag{6.52}
$$

For the isotropic Schwarzschild black holes (Equations 4.5 and 4.15), the velocity field should just consist of u^0 and no spatial component so that the matter is comoving with the expansion of the spatial co-ordinates, since $G^0_0 = \kappa\mu$ and $G^1_1 = -\kappa p$ suggest that $u^0 u_0 = -1$ and $u^1 u_1 = 0$. Calculating $(u^0)^2$ with Equation 6.48 yields something of the form 0/0, so $u^1 = 0$ must be verified by other means to allow u^0 to then be calculated via Equation 6.46. To prove $u^1 = 0$, $T^0_0 = -\mu$ can be multiplied by $u^1 u_1$

$$(\mu + p)u^0 u_0 u^1 u_1 + pu^1 u_1 + q^0 u_0 u^1 u_1 + u^0 q_0 u^1 u_1 = -\mu u^1 u_1 \tag{6.53}$$

to show that

$$pu^1 u_1 = -\mu u^1 u_1, \tag{6.54}$$

but it isn't the case that $p = -\mu$ for either the isotropic constant-mass or M^{c}Vittie cosmological black holes, so $u^1 = g^{11} u_1 = 0$, which means the velocity field must just have a u^0 component. Then u^0 can be calculated using Equation 6.46 to yield

$$u^0 = \pm\sqrt{\frac{-\alpha}{\beta}}. \tag{6.55}$$

For the isotropic constant-mass Schwarzschild black holes (Equation 4.5), Equation 6.55 yields

$$u^0 = \pm\frac{2r + m}{2r - m}, \tag{6.56}$$

and for the M^{c}Vittie black holes (Equation 4.15), Equation 6.55 yields

$$u^0 = \pm\frac{2rR + m}{2rR - m}. \tag{6.57}$$

6.4 Shear and Acceleration

The shear tensor is given by

$$\sigma_{ab} = u_{(a||b)} - \frac{1}{3}\theta h_{ab} + a_{(a} u_{b)}, \tag{6.58}$$

where

$$u_{(a||b)} = \frac{1}{2}(u_{a|b} + u_{b|a}) - \Gamma^c_{ab} u_c, \tag{6.59}$$

$$\theta = u^a{}_{||a} = u^a{}_{|a} + \Gamma^a_{ac} u^c, \tag{6.60}$$

$$h_{ab} = g_{ab} + u_a u_b, \tag{6.61}$$

and

$$a_a = u_{a||c} u^c = (u_{a|c} - \Gamma^d_{ac} u_d) u^c. \tag{6.62}$$

For all cases of the cosmological black holes, there are only u^0, u^1, g_{00}, g_{01}, g_{10}, g_{11}, g_{22}, and g_{33} terms, so the shear tensor looks like

$$\sigma_{00} = u_{0|0} - (\Gamma^0_{00}u_0 + \Gamma^1_{00}u_1) - \frac{1}{3}(u^0{}_{|0} + u^1{}_{|1} + \Gamma^a_{a0}u^0 + \Gamma^a_{a1}u^1)(g_{00} + u_0u_0) + (u_{0|0} - \Gamma^0_{00}u_0)u^0u_0$$

$$\sigma_{01} = \sigma_{10} = \frac{1}{2}(u_{0|1} + u_{1|0}) - (\Gamma^0_{01}u_0 + \Gamma^1_{01}u_1) - \frac{1}{3}(u^0{}_{|0} + u^1{}_{|1} + \Gamma^a_{a0}u^0 + \Gamma^a_{a1}u^1)(g_{01} + u_0u_1)$$
$$+\frac{1}{2}((u_{0|1} - \Gamma^1_{01}u_1)u^1u_1 + (u_{1|0} - \Gamma^0_{10}u_0)u^0u_0)$$
$$\sigma_{02} = \sigma_{20} = -(\Gamma^0_{02}u_0 + \Gamma^1_{02}u_1) - \frac{1}{2}\Gamma^0_{20}u_0u^0u_0$$
$$\sigma_{03} = \sigma_{30} = -(\Gamma^0_{03}u_0 + \Gamma^1_{03}u_1) - \frac{1}{2}\Gamma^0_{30}u_0u^0u_0$$
$$\sigma_{11} = u_{1|1} - (\Gamma^0_{11}u_0 + \Gamma^1_{11}u_1) - \frac{1}{3}(u^0{}_{|0} + u^1{}_{|1} + \Gamma^a_{a0}u^0 + \Gamma^a_{a1}u^1)(g_{11} + u_1u_1) + (u_{1|1} - \Gamma^1_{11}u_1)u^1u_1$$
$$\sigma_{12} = \sigma_{21} = -(\Gamma^0_{12}u_0 + \Gamma^1_{12}u_1) - \frac{1}{2}\Gamma^1_{21}u_1u^1u_1$$
$$\sigma_{13} = \sigma_{31} = -(\Gamma^0_{13}u_0 + \Gamma^1_{13}u_1) - \frac{1}{2}\Gamma^1_{31}u_1u^1u_1$$
$$\sigma_{22} = -(\Gamma^0_{22}u_0 + \Gamma^1_{22}u_1) - \frac{1}{3}(u^0{}_{|0} + u^1{}_{|1} + \Gamma^a_{a0}u^0 + \Gamma^a_{a1}u^1)g_{22}$$
$$\sigma_{23} = \sigma_{32} = -(\Gamma^0_{23}u_0 + \Gamma^1_{23}u_1)$$
$$\sigma_{33} = -(\Gamma^0_{33}u_0 + \Gamma^1_{33}u_1) - \frac{1}{3}(u^0{}_{|0} + u^1{}_{|1} + \Gamma^a_{a0}u^0 + \Gamma^a_{a1}u^1)g_{33}. \qquad (6.63)$$

For the Kerr-Schild cosmological black holes, cancelling out the metric connections (cf. Equations 5.11 and 5.15) that are zero reduces the shear tensor to

$$\sigma_{00} = u_{0|0} - (\Gamma^0_{00}u_0 + \Gamma^1_{00}u_1) - \frac{1}{3}(u^0{}_{|0} + u^1{}_{|1} + \Gamma^a_{a0}u^0 + \Gamma^a_{a1}u^1)(g_{00} + u_0u_0) + (u_{0|0} - \Gamma^0_{00}u_0)u^0u_0$$

$$\sigma_{01} = \sigma_{10} = \frac{1}{2}(u_{0|1} + u_{1|0}) - (\Gamma^0_{01}u_0 + \Gamma^1_{01}u_1) - \frac{1}{3}(u^0{}_{|0} + u^1{}_{|1} + \Gamma^a_{a0}u^0 + \Gamma^a_{a1}u^1)(g_{01} + u_0u_1)$$
$$+\frac{1}{2}((u_{0|1} - \Gamma^1_{01}u_1)u^1u_1 + (u_{1|0} - \Gamma^0_{10}u_0)u^0u_0)$$
$$\sigma_{11} = u_{1|1} - (\Gamma^0_{11}u_0 + \Gamma^1_{11}u_1) - \frac{1}{3}(u^0{}_{|0} + u^1{}_{|1} + \Gamma^a_{a0}u^0 + \Gamma^a_{a1}u^1)(g_{11} + u_1u_1) + (u_{1|1} - \Gamma^1_{11}u_1)u^1u_1$$
$$\sigma_{22} = -(\Gamma^0_{22}u_0 + \Gamma^1_{22}u_1) - \frac{1}{3}(u^0{}_{|0} + u^1{}_{|1} + \Gamma^a_{a0}u^0 + \Gamma^a_{a1}u^1)g_{22}$$
$$\sigma_{33} = -(\Gamma^0_{33}u_0 + \Gamma^1_{33}u_1) - \frac{1}{3}(u^0{}_{|0} + u^1{}_{|1} + \Gamma^a_{a0}u^0 + \Gamma^a_{a1}u^1)g_{33}, \qquad (6.64)$$

where the expansion is

$$\theta = u^0{}_{|0} + u^1{}_{|1} + \Gamma^a_{a0}u^0 + \Gamma^a_{a1}u^1 = u^0{}_{|0} + u^1{}_{|1} + \left(\frac{2\dot{R}}{R}\right)u^0 + \left(\frac{2}{r}\right)u^1. \tag{6.65}$$

These shear matrices are of the form

$$\begin{pmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & e \end{pmatrix}, \tag{6.66}$$

which can be diagonalized as

$$\begin{pmatrix} \frac{a+b\pm\sqrt{(a-b)^2+4c^2}}{2} & 0 & 0 & 0 \\ 0 & \frac{a+b\mp\sqrt{(a-b)^2+4c^2}}{2} & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & e \end{pmatrix}. \tag{6.67}$$

Since the expressions for the velocity field aren't simple, the terms in the shear matrix are extremely long and cannot be reproduced here. However, by substituting numerical values for m, r, and t and tweaking them, it can be verified that the terms in the shear matrix are non-zero and are not due to numerical noise, so shear does in fact exist. Likewise, it can be verified that the acceleration is also non-zero.

For the isotropic black holes, there are no u^1, g_{01}, or g_{10} terms, and the non-zero metric connections (cf. Equations 5.18 and 5.22) are the same components as for the Kerr-Schild black holes (other than that $\Gamma^0_{00} = 0$ for the constant-mass isotropic black holes), so this reduces the shear tensor to

$$\begin{aligned}
\sigma_{00} &= u_{0|0} - \Gamma^0_{00}u_0 - \frac{1}{3}(u^0{}_{|0} + \Gamma^a_{a0}u^0)(g_{00} + u_0u_0) + (u_{0|0} - \Gamma^0_{00}u_0)u^0u_0 \\
\sigma_{01} = \sigma_{10} &= \frac{1}{2}u_{0|1} - \Gamma^0_{01}u_0 - \frac{1}{2}\Gamma^0_{10}u_0u^0u_0 \\
\sigma_{11} &= -\Gamma^0_{11}u_0 - \frac{1}{3}(u^0{}_{|0} + \Gamma^a_{a0}u^0)g_{11} \\
\sigma_{22} &= -\Gamma^0_{22}u_0 - \frac{1}{3}(u^0{}_{|0} + \Gamma^a_{a0}u^0)g_{22} \\
\sigma_{33} &= -\Gamma^0_{33}u_0 - \frac{1}{3}(u^0{}_{|0} + \Gamma^a_{a0}u^0)g_{33},
\end{aligned} \tag{6.68}$$

where the expansion (in the case of zero charge) is

$$\theta = u^0{}_{|0} + \Gamma^a_{a0}u^0 = 0 + 3\frac{\dot{R}}{R}\frac{2r+m}{2r-m} \tag{6.69}$$

for the constant-mass isotropic black holes and

$$\theta = u^0{}_{|0} + \Gamma^a_{a0}u^0$$
$$= -\frac{4mr\dot{R}}{(2Rr-m)^2} + \left(\frac{4\dot{R}mr}{(2Rr-m)(2rR+m)} + \frac{3(2Rr-m)\dot{R}}{(2Rr+m)R}\right)\frac{2Rr+m}{2Rr-m} = 3\frac{\dot{R}}{R} \tag{6.70}$$

for the M^cVittie black holes. Unlike the volume expansion calculated in terms of the co-ordinate volume and time (Equations 6.24 and 6.34), the volume expansion appears to be changed for the constant-mass black holes and not for the M^cVittie black holes. Since

$$\theta = \frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\tau}, \tag{6.71}$$

where τ is the proper time, and

$$\frac{\dot{V}}{V} = \frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t_c} \tag{6.72}$$

and

$$u^0 = \frac{\mathrm{d}t_c}{\mathrm{d}\tau}, \tag{6.73}$$

then the different calculations of the volume expansion are related by

$$\theta = \frac{\dot{V}}{V}u^0, \tag{6.74}$$

so Equations 6.69 and 6.70 are in fact consistent with the previous calculations of $\dot{V}/V = 3\dot{R}/R$ for the constant-mass black holes and

$$\frac{\dot{V}}{V} = 3\frac{\dot{R}}{R}\left(1 - \frac{m}{rR}\right) \approx 3\frac{\dot{R}}{R}\left(\frac{2rR-m}{2rR+m}\right) \tag{6.75}$$

for a thin shell as a function of r for the M^cVittie black holes.

Since Raychaudhuri's equation (Equation 1.1) is for $\theta_{|a}u^a$, then this is

$$\dot{\theta}u^0 = \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = 3\left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R}\right)\left(\frac{2r+m}{2r-m}\right)^2 \tag{6.76}$$

for the constant-mass black holes and

$$\dot{\theta}u^0 = \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = 3\left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R}\right)\left(\frac{2rR+m}{2rR-m}\right) \tag{6.77}$$

for the M^cVittie black holes. Compared with

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = 3\left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R}\right) \tag{6.78}$$

for the standard FRW universe—which is negative—then the modification due to the cosmological black holes will lead to a greater slowing of the expansion than that in an FRW universe (and in the case of the M^cVittie black holes will lead to a change of sign inside $r = m/(2R)$ so that the expansion actually increases there rather than decreases, most likely due to the pressure being negative there). The extra decrease in the volume expansion may be due to either excess mass-energy, shear, or deceleration.

For the constant-mass isotropic black holes, Γ^0_{00} and ${u^0}_{|0}$ are both zero, so the shear tensor components (in the case of zero charge) are

$$\sigma_{00} = -\frac{1}{3}\theta(g_{00} + u_0 u_0) = -\frac{\dot{R}}{R}\frac{2r+m}{2r-m}\left(-\frac{(1-\frac{m}{2r})^2}{(1+\frac{m}{2r})^2} + \frac{(2r-m)^2}{(2r+m)^2}\right) = 0$$

$$\sigma_{01} = \sigma_{10} = \frac{1}{2}u_{0|1} - \frac{1}{2}\Gamma^0_{01}u_0 = \frac{-4m}{2(2r+m)^2} - \frac{4m}{2(2r+m)(2r-m)}\left(-\frac{2r-m}{2r+m}\right) = 0$$

$$\sigma_{11} = -\Gamma^0_{11}u_0 - \frac{1}{3}\theta g_{11} = -\frac{(2r+m)^6}{16r^4(2r-m)^2}\left(-\frac{2r-m}{2r+m}\right)\dot{R}R - \frac{\dot{R}}{R}\frac{2r+m}{2r-m}R^2\left(1+\frac{m}{2r}\right)^4 = 0$$

$$\sigma_{22} = -\Gamma^0_{22}u_0 - \frac{1}{3}\theta g_{22} = r^2\sigma_{11} = 0$$

$$\sigma_{33} = -\Gamma^0_{33}u_0 - \frac{1}{3}\theta g_{33} = r^2\sin^2\theta\sigma_{11} = 0. \tag{6.79}$$

For the M^cVittie black holes the shear tensor components are

$$\sigma_{00} = u_{0|0} - \Gamma^0_{00}u_0 - \frac{1}{3}\theta(g_{00} + u_0 u_0) = \frac{-4rm\dot{R}}{(2Rr+m)^2} -$$

$$\frac{4rm\dot{R}}{(2Rr+m)(2Rr-m)}\left(-\frac{2rR-m}{2rR+m}\right) - \frac{\dot{R}}{R}\left(-\frac{\left(1-\frac{m}{2Rr}\right)^2}{\left(1+\frac{m}{2Rr}\right)^2} + \frac{(2Rr-m)^2}{(2Rr+m)^2}\right) = 0$$

$$\sigma_{01} = \sigma_{10} = \frac{1}{2}u_{0|1} - \frac{1}{2}\Gamma^0_{01}u_0 = \frac{-4mR}{2(2Rr+m)^2} - \frac{4mR}{2(2Rr+m)(2Rr-m)}\left(-\frac{2r-m}{2r+m}\right) = 0$$

$$\sigma_{11} = -\Gamma^0_{11}u_0 - \frac{1}{3}\theta g_{11} = -\frac{(2Rr+m)^5\dot{R}}{16r^4R^3(2Rr-m)}\left(-\frac{2r-m}{2r+m}\right) - \frac{\dot{R}}{R}R^2\left(1+\frac{m}{2rR}\right)^4 = 0$$

$$\sigma_{22} = -\Gamma^0_{22}u_0 - \frac{1}{3}\theta g_{22} = r^2\sigma_{11} = 0$$

$$\sigma_{33} = -\Gamma^0_{22} u_0 - \frac{1}{3}\theta g_{22} = r^2 \sin^2\theta \sigma_{11} = 0. \tag{6.80}$$

Thus, the isotropic cosmological black holes are shear-free, which isn't surprising considering the metrics are composed in terms of isotropic spatial co-ordinates. They do however have radial acceleration

$$a^1 = \Gamma^1_{00}(u^0)^2, \tag{6.81}$$

which is

$$a^1 = -\frac{64mr^4}{R^2(2r+m)^5(2r-m)} \tag{6.82}$$

in the case of the constant-mass black holes and

$$a^1 = \frac{64mr^4R^3}{(2rR+m)^5(2rR-m)} \tag{6.83}$$

in the case of the M$^{\text{c}}$Vittie black holes. Note that the acceleration has opposite signs for the different cosmological black holes, which might be related to the greater decrease in the volume expansion for the constant-mass black holes than for the M$^{\text{c}}$Vittie black holes (cf. Equations 6.76 and 6.77). The acceleration is most likely due to the pressure gradient exerting a force on the mass-energy.

Chapter 7

Summary and Discussion

New cosmological black hole solutions were obtained in this thesis by generalizing the expanding Kerr-Schild cosmological black holes to obtain the charged case (Equations 3.26 and 3.53), by performing a Kerr-Schild transformation of the Einstein-de Sitter universe instead of a closed universe to obtain non-expanding Kerr-Schild cosmological black holes in asymptotically-flat universes (Equations 3.16, 3.33, 3.47, and 3.57), and by performing a conformal transformation on the isotropic forms of the metric for isolated black holes to obtain cosmological black holes that have completely physical spacetimes (Equations 4.5 and 4.28). The expanding Kerr-Schild Schwarzschild black holes were studied more comprehensively to specifically yield the black hole case (Equation 3.40) instead of just the white hole, and also for the case of a radiation-dominated background universe (Equation 3.3) instead of just dust. All the Kerr-Schild black holes, as well as the M$^{\text{c}}$Vittie (Equation 4.15) and Gao & Zhang (Equation 4.36) black holes were examined to see where they are physical and are solutions of Einstein's Field Equations.

While the inclusion of charge generally doesn't significantly change the metric from that of a Schwarzschild black hole, simply knowing that the exact solutions exist is worthwhile, since there are no exact solutions for rotating cosmological black holes for instance. Also, the non-expanding Kerr-Schild cosmological black holes previously obtained were only in closed universes—it is possible to generalize these metrics to flat or open universes by setting the radius of curvature to infinity or multiplying it by i (as noted by Krasiński 1997); however, that still leaves the problem of physically interpreting the energy-momentum tensor before solutions can actually be said to exist for these cases. Thus, the flat cases were interpreted to yield solutions.

Eardley (1974) found that white holes are unstable and will be converted into black holes (which could have homogenized the early universe), so the cosmological white holes presented in Chapter 3 may not be physically relevant. However, FRW is itself unstable to perturbations, so even the cosmological black hole solutions are probably unstable. Thus, stability was not an issue considered in this thesis, although in the future it would be interesting to examine the influence of instabilities in the background universe on black holes. It also may not be that likely to form Reissner-Nordström black holes, although no explanation of their formation is needed any more so than for Schwarzschild black holes or a perfectly homogeneous Big Bang if they just happen to exist from the moment of the Big Bang, but quantum mechanics should allow for Reissner-Nordström black holes to destroy themselves much more easily than Schwarzschild black holes, so it would be unlikely they would remain around long. However, no adequate theory of quantum gravity exists, so this thesis was written simply within the context of General Relativity and such issues were not considered.

Since expanding Kerr-Schild cosmological black holes are obtained by performing a conformal transformation on isolated black hole spacetimes in Kerr-Schild form, it might not be expected that performing the conformal transformation on isolated black hole spacetimes in isotropic form would lead to a physically different solution being as the different forms of the isolated black hole spacetimes are all simply related by co-ordinate transformations. Yet whereas all other cosmological black hole solutions (aside from Swiss cheese black holes) violate the energy conditions in some region of spacetime, the isotropic expanding cosmological black holes were found to have a physical energy-momentum tensor throughout spacetime. Although the pressure (in the radiation-dominated case) and density become infinite at the event horizon, the energy conditions are never violated, and it might not be that unrealistic to have infinite pressure and density, since the universe would have began that way and would simply need to maintain the original density and pressure at the event horizon. The difference in outcome with the conformal transformation is more than can be explained by the original co-ordinate transformation between different forms of the black hole metrics, as a co-ordinate transformation to the resultant spacetime can't make the unphysical spacetime look like the physical one. Thus, it must be the act of performing the conformal transformation and the non-conformally-invariant nature of the energy-momentum tensor that makes it possible to bring about completely different physical scenarios depending on the original form of the black hole metric.

It is interesting to note that while Thakurta (1981) performed a conformal transformation on the Boyer-Lindquist form of the Kerr metric (which without rotation is the standard Schwarzschild form, Equation 2.15), Sultana (2003) performed a conformal transformation on the Eddington-Finkelstein (Kerr-Schild) form of the Schwarzschild metric. Had Thakurta interpreted the energy-momentum tensor to study the density and pressure (rather than only so far as to say it is a perfect fluid with heat conduction, which in that respect is the same result as Sultana's), the pressure and density would have been found to be negative inside the event horizon (much like the McVittie solution), so in fact the very structure of interest in the spacetime isn't a valid part of the spacetime.

Generally, people appear to have been somewhat careless as far as actually interpreting energy-momentum tensors and seeing whether they are physical, being more content to simply examine the metrics and be satisfied that they look like the superposition of an FRW universe and a black hole. Without the existence of a physical energy-momentum tensor, a metric doesn't correspond to anything physically possible, so it is fruitless to study metrics that have no corresponding mass-energy distribution that could give rise to them, and it isn't valid to refer to them as "solutions"; otherwise, any random metric should be called a solution, since any random metric will yield an energy-momentum tensor, just not necessarily a physical one.

If people wish to speak about the cosmological black holes as being models of primordial black holes that could exist in our own universe, it is clearly necessary to have solutions for cosmological black holes that evolve from being in radiation-dominated to matter-dominated background universes since the universe hasn't been matter-dominated forever. Thus, in this thesis it was shown that it is possible to match radiation-dominated and matter-dominated Einstein-de Sitter universes directly together across a hypersurface of constant time. This isn't the most realistic model of a universe that evolves from radiation domination to matter domination, but it does provide a simple means to create primordial black hole solutions by matching the cosmological black holes in radiation-dominated backgrounds to cosmological black holes in matter-dominated backgrounds. Unrelated to the problem of cosmological black holes, while it is possible to obtain better models of homogeneous cosmological models that gradually evolve from radiation-domination to matter-domination, it is still interesting to know that radiation-dominated and matter-dominated universes can be directly matched, and the matching of a matter-dominated universe onto a subsequently radiation-dominated universe would be useful as a realistic model for a universe full of matter and antimatter that suddenly annihilates to leave radiation.

The primary motivation behind this thesis was to see how cosmological inhomogeneities can influence the expansion of the universe, not any specific interest in black holes themselves. In fact, based on the considerations of event horizons in this thesis, the Vaidya-type and McVittie-type cosmological black holes may not actually be black holes in the conventional sense: these spacetimes are not just conformal transformations of black holes, so they don't have to preserve the causal structure at their event horizons, and the shrinkage of these surfaces in comoving co-ordinates appears to allow photons to actually escape. The cosmological black holes do make good models of inhomogeneities though because they are simple exact solutions, yet unlike the isolated black hole spacetimes that contain any mass, charge, or rotation at a singularity in the spacetime such that the energy-momentum tensor is simply a vacuum, the cosmological black holes actually modify the background universe so that the density and pressure need not be homogeneous and heat conduction may be introduced. While the cosmological black holes may be obtained from the simple spacetimes for isolated black holes, the cosmological black hole spacetimes don't merely introduce a singularity in an otherwise homogeneous universe. This makes them interesting models of cosmological inhomogeneities.

The cosmological black holes were found to decrease the universe's volume expansion, which must either be due to excess mass-energy associated with them, shear they introduce, or deceleration (from a pressure gradient). Since the influence of the cosmological black holes only falls off asymptotically with radius (unlike Swiss cheese black holes) exact solutions for universes with more than one black hole would be practically impossible to devise; however, it seems intuitive that a universe containing many such inhomogeneities would contain many such volumes of decreased expansion. While a single black hole in an infinite universe has no effect on the volume expansion of the universe as a whole, if a universe were composed of a network of volumes of decreased expansion, then the volume expansion of the universe as a whole could be impacted. Also, it isn't just the existence of excess mass-energy in the inhomogeneity that decreases the volume expansion: the fact that shear or deceleration exists in certain cases means that according to Raychaudhuri's equation the volume expansion of the universe is decreased even without any extra mass-energy in the inhomogeneity. Thus, for a more realistic model of an inhomogeneous universe that contained several overdensities and underdensities, even if the redistribution of mass-energy were such that it had no overall impact on the universe's expansion, and even if acceleration tended to cancel deceleration, any shear due to the inhomogeneities would always tend to decrease the universe's expansion.

Much of the previous work examining the impact of inhomogeneities on the universe's expansion has attempted to show that the backreaction of inhomogeneities on the background universe would actually lead to the universe accelerating; however, much of this work appears to have been biased toward trying to explain the cosmological constant as being due to the universe's structure formation. Also, this work depends on spatial-averaging approximations, which depending on how they are handled, may lead to completely different results, so it doesn't yield exact results that can be easily trusted. Raychaudhuri's equation applies even in the Newtonian case though, so shear will decrease the volume expansion regardless of the spatial-averaging problem in General Relativity. Thus, it seems likely that shear (assuming vorticity and acceleration are insignificant) should be more significant than any purely relativistic effects due to spatial averaging, so that inhomogeneities should generally be expected to decrease the universe's volume expansion.

Assuming inhomogeneities wouldn't tend to introduce acceleration in the velocity field (with regions of acceleration only being relevant locally for pressure-supporting inhomogeneities against collapse), if the expansion of the universe were to be accelerated, then according to Raychaudhuri's equation, it would have to be due to vorticity, yet vorticity is generally assumed to be zero in studying the backreaction of inhomogeneities on the universe. If the expansion of our universe is accelerating, and if this acceleration isn't due to a cosmological constant, vorticity would seem to be the best explanation, although it appears that rotation in the universe only operates at the level of individual objects to prevent them from collapsing, rather than there being any rotation of the universe as a whole. For the sake of speculation, if frame dragging due to the Lense-Thirring effect were to cause the universe to be assigned some slight rotation from each rotating object within it, then macroscopically it could behave like it had some slight rotation. This can be understood by comparing with the Gödel (1949) universe. In the Gödel universe, the individual rotation of the matter at each point in the manifold is able to maintain the universe against collapse because the inertial compass rotates along with each point such that inertially the matter at each point is at rest and sees the rest of the universe revolving around it. Normally when we think of rotation, we think of an object rotating with respect to the inertial compass without affecting it, but according to General Relativity, frame dragging causes the inertial compass to be rotated slightly in the sense of a rotating object. Since the complete rotation of the inertial compass along with the matter at each point in the Gödel universe is able to give rise to the universe supporting itself with rotation, then the slight rotation of the inertial compass along with each of the rotating objects in our universe could conceivably give rise to the universe being partially rotationally supported.

It is interesting that no one has been able to obtain solutions for Kerr cosmological black holes. As discussed by Thakurta (1981), this isn't surprising since no one has obtained an exact solution for a Kerr interior either. A solution exists only for an isolated rotating ring singularity, not an extended rotating object. Because the ring singularity isn't a part of the spacetime, in the sense that its mass and angular momentum don't show up in the energy-momentum tensor, the Kerr solution isn't really a solution for a rotating object. Trying to obtain a cosmological Kerr black hole is similar to the problem of trying to find a Kerr interior solution for a rotating body, since embedding a Kerr black hole in a cosmological model would require it to swirl the surrounding mass-energy via the Lense-Thirring effect. Either the physics involved in rotation is simply too complicated (tending to deform a sphere into a Maclaurin spheroid etc.) for us to interpret the energy-momentum tensors, or General Relativity may not be compatible with the existence of absolute rotation. The only real solutions with rotation are homogeneous spacetimes like the Gödel universe, yet in the Gödel universe the matter at every point is individually in rotation relative to the matter at other points, so there really is no absolute rotation (although due to the fact that the inertial compass rotates along with the matter at each point, every inertial observer will infer that the rest of the universe is in absolute rotation about it). While the existence of absolute rotation is generally assumed, it should be kept in mind that this may not be valid.

To summarize, several new cosmological black hole solutions have been obtained, most notably the expanding isotropic black holes, which have completely physical energy-momentum tensors. Previous solutions for Kerr-Schild and isotropic cosmological black holes have been interpreted more carefully and analyzed to determine what regions of spacetime satisfy the energy conditions and can be deemed valid. Primordial black holes have been obtained via direct matchings of radiation-dominated cosmological black hole backgrounds onto dust backgrounds. Finally, it has been concluded that the most likely impact of inhomogeneities in a universe with overdense and underdense regions would be to introduce shear, increasing the deceleration of the universe's expansion compared with that of a homogeneous universe.

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