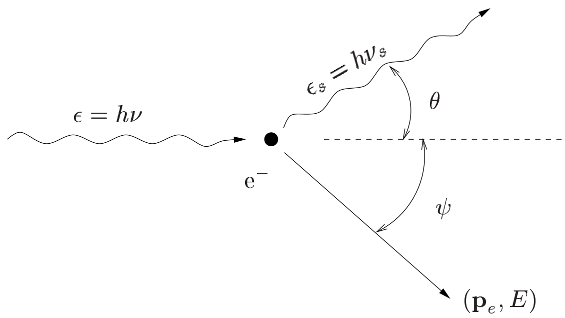
5. Compton and Inverse Compton Scattering

5.1 Compton Scattering

This is the generalization of Thomson scattering to high frequencies ($h\nu > mc^2$) but still nonrelativistic velocities. We know that in this regime, the electron recoils and the scattering is noncoherent. There are *three* unknowns here for a chosen θ: the energy of the scattered photon ϵ_s, the angle characterizing the electron's motion ψ, and the recoil velocity of the electron v_e. Conservation of energy and the two components of momentum determine the following solution:



$$\epsilon_s \;=\; \frac{\epsilon}{1 + (\epsilon/mc^2)\,(1 - \cos\theta)} \;\leq\; \epsilon \;.$$

Thus, the photon is always *red-shifted* because the recoil of the electron carries away some of the incident photon's energy. This in *noncoherent* scattering. Note that if $\epsilon \ll mc^2$, we recover Thomson scattering for which $\sigma = \sigma_{\rm T} =$ constant.

In the general case, however, the cross-section is energy or frequency dependent. The differential cross-section is given by the Klein-Nishina formula derived from quantum electrodynamics:

$$\frac{d\sigma}{d\Omega} = \frac{r_o^2}{2}\left(\frac{\epsilon}{\epsilon_s} + \frac{\epsilon_s}{\epsilon} - \sin^2\theta\right)\frac{\epsilon_s^2}{\epsilon^2} \quad\rightarrow\quad \frac{r_o^2}{2}\,(1 + \cos^2\theta) \quad (\epsilon = \epsilon_s)$$

which assumes unpolarized incident radiation. This relation reduces to the Thomson form (that we considered earlier) when $\epsilon_s = \epsilon$. Clearly, the differential cross-section drops like ϵ^{-1} for large ϵ. Hence, scattering becomes less and less efficient at high photon energies which is intuitively what we should expect. Note that the classical theory fails here because it does not guarantee the conservation of energy and momentum.

The *total* cross-section for Compton scattering, σ_e (or $\sigma_{\rm C}$) to distinguish it from $\sigma_{\rm T}$, is a rather complicated function of ϵ:

$$\sigma_e(\alpha) = \frac{3}{4}\sigma_{\rm T}\left\{\frac{(1+\alpha)}{\alpha^3}\left[\frac{2\alpha(1+\alpha)}{(1+2\alpha)} - \ln(1+2\alpha)\right] + \frac{\ln(1+2\alpha)}{2\alpha} - \frac{(1+3\alpha)}{(1+2\alpha)^2}\right\}$$

where $\alpha \equiv \epsilon/mc^2 = h\nu/mc^2$. This parameter is of order unity when $\lambda \sim 0.02$Å or when $T \sim 10^9\,{}^\circ$K.

$$\sigma_e(\alpha) \longrightarrow \begin{cases} \sigma_{\rm T}\left(1 - 2\alpha + \frac{26}{5}\alpha^2 + \ldots\right) & (\alpha \rightarrow 0) \\ \dfrac{3\sigma_{\rm T}}{8\alpha}\left(\ln 2\alpha + \dfrac{1}{2}\right) & (\alpha \rightarrow \infty) \end{cases}$$

The Compton cross-section σ_e has the same limiting ϵ^{-1} behavior as $d\sigma/d\Omega$ at large incident photon energies. Given the Compton cross-section, the scattered power from a non-relativistic electron can be written down directly:

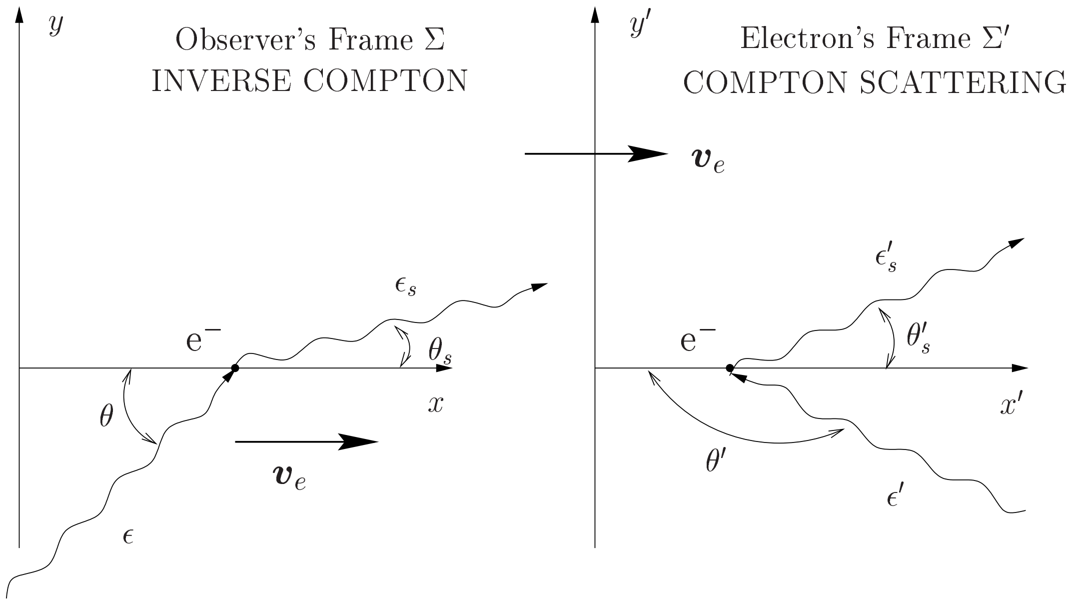
$$P_{\rm C}(\nu) = \sigma_e(\nu)\,c\,U_\nu \quad (\gamma \rightarrow 1) \;.$$

This equation assumes that the kinetic energy of the electron is much less than the photon energy; i.e., $E_e \equiv \frac{1}{2}mv_e^2 \ll h\nu$.

5.2 Inverse Compton Scattering

In this case, we allow for ultra-relativistic electrons which can transfer some of their kinetic energy to photons. That is, the scattered photons are *blue*-shifted instead of being red-shifted as they are when energetic photons scatter off nonrelativistic electrons.

Consider the geometry in both the observer's frame (Σ) and the electron's frame (Σ'):



To solve this problem, we transform to the moving (electron's) frame where the Compton effect applies and then back to the observer's frame. In each case, we make the appropriate relativistic Doppler transformation.

We want ϵ_s in terms of ϵ. The order of the transformations is as follows:

$$\epsilon \quad \overset{\text{Doppler}}{\longrightarrow} \quad \epsilon' \quad \overset{\text{Compton}}{\longrightarrow} \quad \epsilon'_s \quad \overset{\text{Doppler}}{\longrightarrow} \quad \epsilon_s \ .$$

(1) First Doppler Transformation:

$$\epsilon' = \gamma\epsilon\left(1 - \frac{v}{c}\cos\theta\right) \qquad \text{(eqn. 4.12a in text)}\,;$$

(2) Compton Scattering:

$$\epsilon'_s = \frac{\epsilon'}{1 + (\epsilon'/mc^2)} \qquad \text{(averaging over } \theta'_s)\,;$$

(3) Second Doppler Transformation:

$$\begin{aligned}
\epsilon_s &= \gamma\epsilon'_s\left(1 + \frac{v}{c}\cos\theta'_s\right) \qquad \text{(eqn. 4.12b in text)} \\
&= \gamma^2\epsilon\left(1 + \frac{\gamma\epsilon}{mc^2}\right)^{-1}\left(1 - \frac{v}{c}\cos\theta\right)\left(1 + \frac{v}{c}\cos\theta'_s\right) \\
&\simeq \gamma^2\epsilon\left(1 + \frac{\gamma\epsilon}{mc^2}\right)^{-1} \quad \longrightarrow \quad \gamma^2\epsilon \quad (\gamma\epsilon \ll mc^2)
\end{aligned}$$

if we assume that θ and θ'_s are randomly distributed; i.e., photon directions are isotropic.

Thus, low energy photons ($\gamma\epsilon \ll mc^2$) can experience a γ^2 boost factor if they scatter off relativistic electrons. This is the essence of inverse Compton scattering. At high energies, however, there is a cap on the scattered photon energy; the latter cannot exceed the energy of the electron which should not be a surprise. Thus,

$$\epsilon_s \to \gamma mc^2 \quad (\gamma\epsilon \gg mc^2) \ .$$

Example: $\gamma = 10^3, \ \nu = 10^{10}\,\text{Hz} \qquad \Longrightarrow \qquad \nu_s = \gamma^2\nu \sim 10^{16}\,\text{Hz}$
which is in the hard UV or soft X-ray frequency range!

What is the condition required for energy to flow from the electrons to the photons? We want $\epsilon_s > \epsilon$; i.e.,

$$\gamma^2 > 1 + \frac{\gamma\epsilon}{mc^2}$$

or

$$(\gamma+1)(\gamma-1)\,mc^2 > \gamma\epsilon$$

or

$$(\gamma-1)\,mc^2 > \frac{\gamma\epsilon}{\gamma+1}$$

or

$$\text{kinetic energy of electrons} > \left(\frac{1}{2}\ \text{to}\ 1\right) \times \text{the photon energy} \ .$$

Therefore, energy is transferred from the particles to the photons when the kinetic energy per particle is comparable to or larger than the energy per photon. Again, this is intuitively what we should expect.

Inverse Compton Scattered Power

To get the scattered power from a single electron, we move to the rest frame of the electron and invoke the cross-section for Compton scattering. Actually, we restrict our attention to relatively low energy photons (even in the moving frame) and use the Thomson scattering coefficient σ_{T}. Thus, in the electron's frame,

$$P'_\nu = \sigma_e(\nu)\,S'_\nu \to \sigma_{\mathrm{T}}\,S'_\nu = \sigma_{\mathrm{T}}\,c\,U'_\nu \quad (\gamma h\nu \ll mc^2).$$

Since σ_{T} is independent of ν, we can write

$$P' = \sigma_{\mathrm{T}}\,c\,{U_{\mathrm{R}}}' = P_{\mathrm{IC}}$$

where ${U_{\mathrm{R}}}'$ is the total energy density in the EM-field of the photons as seen by the electron.

But we want to express ${U_{\mathrm{R}}}'$ in terms of U_{R}. This is easy to do if we invoke the Lorentz transformation rules for $\boldsymbol{E}$ and $\boldsymbol{B}$-fields (see text, p128). In the electron's rest frame, the components of $\boldsymbol{E}$ perpendicular and parallel to the velocity $\boldsymbol{v}$ are

$$\boldsymbol{E}'_\perp = \gamma\,[\boldsymbol{E}_\perp + (\boldsymbol{\beta}\times\boldsymbol{B})] \ , \qquad \boldsymbol{E}'_\parallel = \boldsymbol{E}_\parallel \ .$$

Clearly, we can ignore $\boldsymbol{E}_{||}$ for large γ and write

$$U_{\rm R}{}' = \frac{\langle|\boldsymbol{E}'|^2\rangle}{4\pi} = \frac{\gamma^2}{4\pi}\left[\langle|\boldsymbol{E}_\perp|^2\rangle + \langle|\boldsymbol{\beta}\times\boldsymbol{B}|^2\rangle + 2\langle\boldsymbol{E}_\perp\cdot(\boldsymbol{\beta}\times\boldsymbol{B})\rangle\right] \qquad (\gamma^2 \gg 1)\ .$$

Consider the second and third terms on the right-hand-side of this equation:

(a) $\langle|\boldsymbol{\beta}\times\boldsymbol{B}|^2\rangle = \langle|\boldsymbol{B}_\perp|^2\rangle$ (because $|\boldsymbol{\beta}| \simeq 1$)

(b) $\langle\boldsymbol{E}_\perp\cdot(\boldsymbol{\beta}\times\boldsymbol{B})\rangle = \langle-\boldsymbol{E}_\perp\cdot(\boldsymbol{B}_\perp\times\boldsymbol{\beta})\rangle = \langle-(\boldsymbol{E}_\perp\times\boldsymbol{B}_\perp)\cdot\boldsymbol{\beta}\rangle = 0$
since the Poynting flux is just as likely to be parallel to $\boldsymbol{\beta}$ as antiparallel for an *isotropic* radiation field.

Therefore,

$$U_{\rm R}{}' = \frac{\gamma^2}{4\pi}\left[\langle|\boldsymbol{E}_\perp|^2\rangle + \langle|\boldsymbol{B}_\perp|^2\rangle\right] = \frac{4}{3}\gamma^2\left[\frac{\langle|\boldsymbol{E}|^2\rangle}{8\pi} + \frac{\langle|\boldsymbol{B}|^2\rangle}{8\pi}\right] = \frac{4}{3}\gamma^2 U_{\rm R}\ .$$

where we have used the fact that $\boldsymbol{E}_\perp$ includes two of the three components of $\boldsymbol{E}$ (and similarly for $\boldsymbol{B}_\perp$). Finally, we have

$$P_{\rm IC} = \frac{4}{3}\,\sigma_{\rm T}\,c\,\gamma^2\,U_{\rm R} \quad (\gamma^2 \gg 1)\ .$$

Comparing inverse Compton and synchrotron losses by a single electron, we see that

$$\frac{P_{\rm IC}}{P_{\rm S}} = \frac{U_{\rm R}}{U_{\rm M}}\ .$$

Note that the expression for $P_{\rm IC}$ can be generalized to arbitrary electron energies by replacing γ^2 with $\gamma^2 - 1 = \gamma^2\beta^2$ (see the text). We did this earlier for the synchrotron power $P_{\rm S}$. In both cases, this substitution makes the power radiated from a low energy electron proportional to v^2 which is proportional to $E_{\rm kin}$. That is, as the electron energy moves from relativistic values to non-relativistic ones, the power dependence on the energy changes from E^2 to E. There is still a restriction on the formulae in that the photon energy in the electron's frame must always be small enough that the Thomson cross-section can be used; i.e., $\gamma\epsilon \ll mc^2$.

Summary

Unfortunately, there is no general formula that covers all cases – that is, arbitrary photon energies (ϵ) and arbitrary electron energies (γ). We have considered here only the most common situations, namely,

(i) Thomson scattering (soft radiation, low energy electrons):

$$P_{\rm T}(\nu) = \sigma_{\rm T}\,c\,U_\nu \qquad (E_e < \epsilon < mc^2)\,.$$

(ii) Compton scattering (hard radiation, low energy electrons):

$$P_{\rm C}(\nu) = \sigma_e(\nu)\,c\,U_\nu \qquad (E_e < mc^2 < \epsilon)\,.$$

(iii) Inverse Compton scattering (soft radiation, arbitrary energy electrons):

$$P_{\rm IC} = \frac{4}{3}\sigma_{\rm T}\,c\,\gamma^2\beta^2\,U_{\rm R} \qquad (\gamma\epsilon < mc^2)\,.$$

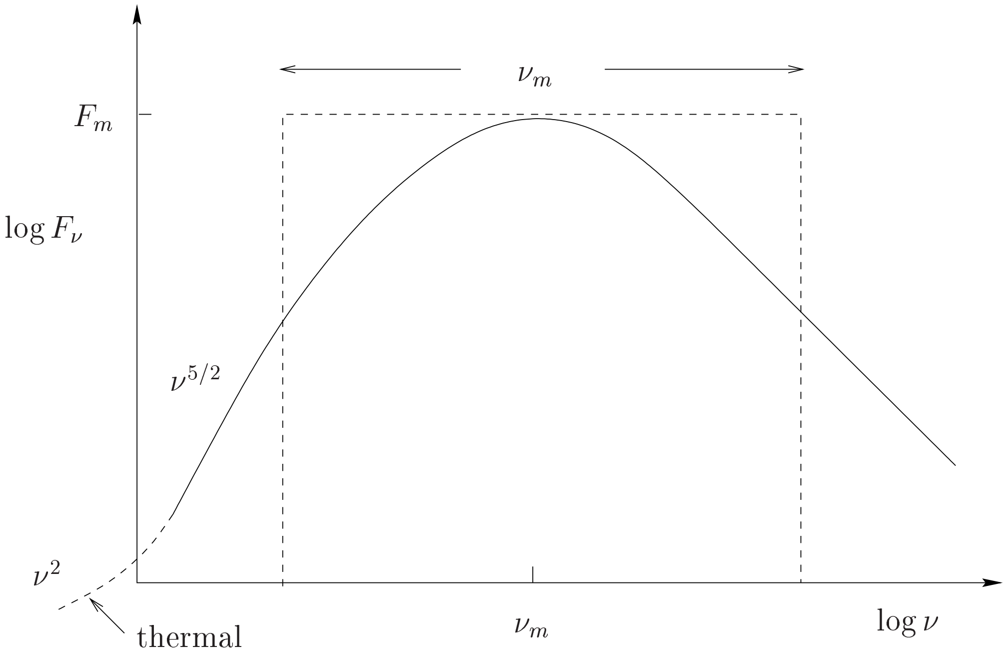
5.3 Maximum Brightness Temperature

The following argument is given by Frank Shu in his text *Radiation* (p186). He calculates the effect of inverse Compton scattering on the spectrum of a synchrotron source. In particular, he shows that IC scattering limits the peak synchrotron flux by blue shifting photons out to higher frequencies.

Consider a uniform spherical source of synchrotron radiation of radius R and distance D. Assume that the radiation field in the source consists primarily of the synchrotron photons. Then, in the optically thin limit for which $U_{\rm R}(R) \sim U_{\rm R}(0)$, we can write

$$S(R) = U_{\rm R} c \simeq \frac{L_s}{4\pi R^2} \quad \Longrightarrow \quad U_{\rm R} \simeq \frac{L_s}{4\pi R^2 c} \simeq \frac{4\pi D^2 F_m \nu_m}{4\pi R^2 c} = \frac{F_m \nu_m}{\theta^2 c}$$

where we approximate the integrated flux by $F_m \nu_m$ as illustrated in the following sketch. Also, $\theta \equiv R/D$ is half the angle subtended by the source. Thus, the *solid* angle subtended is $\pi\theta^2$. [In the foregoing equation, don't confuse the subscript 'R' which indicates radiation energy density and the variable 'R' which refers to the radius of the source.]



Substitute for F_m in terms of the brightness temperature:

$$F_m = I_m \, d\Omega \propto T_{bm} \nu_m^2 \theta^2 \qquad \text{(Rayleigh-Jeans law for a black-body)} .$$

Eliminate ν_m using the surface brightness:

$$F_m \propto \theta^2 \left(\frac{j_\nu}{\alpha_\nu} \right) \propto \theta^2 B^{-1/2} \nu_m^{5/2} \qquad \text{(optically thick)} .$$

Take the $\frac{4}{5}$th power of each side to get

$$\nu_m^2 \propto F_m^{4/5} B^{2/5} \theta^{-8/5} .$$

Then,

$$F_m \propto T_{bm} \left(F_m^{4/5} B^{2/5} \theta^{-8/5} \right) \theta^2$$

or

$$F_m^{1/5} \propto T_{bm} B^{2/5} \theta^{2/5}$$

or

$$F_m \propto T_{bm}^5 B^2 \theta^2 \ .$$

Now, we can compare synchrotron and inverse Compton losses:

$$\frac{P_{\rm IC}}{P_{\rm S}} = \frac{U_{\rm R}}{U_{\rm M}} = \frac{F_m \nu_m}{\theta^2 c} \left(\frac{8\pi}{B^2} \right) \propto \frac{T_{bm}^5 B^2 \theta^2 \nu_m}{\theta^2 c} \left(\frac{8\pi}{B^2} \right) \propto T_{bm}^5 \nu_m \ .$$

Working out the constant of proportionality, we find

$$\frac{P_{\rm IC}}{P_{\rm S}} \simeq \left(\frac{T_{bm}}{10^{12}\,^\circ\mathrm{K}} \right)^5 \left(\frac{\nu_m}{10^{8.5}\,\mathrm{Hz}} \right) \ .$$

This shows that the ratio of inverse Compton power to synchrotron power is very small until T_b reaches $10^{12}\,^\circ$K whereupon the IC losses become *catastrophic*! The "lost" energy goes into blue-shifting the photons around ν_m out to much higher frequencies where T_b is low. Thus, T_{bm} can never rise above $10^{12}\,^\circ$K, at least for steady-state sources. What happens is that the synchrotron spectrum is distorted in such a way that photons near the peak (where the flux is high) are shifted out to higher frequencies (where the flux and brightness temperature are low).

For nonequilibrium systems (e.g., variable quasars), T_b could rise as high as $6 \times 10^{15}\,^\circ$K for brief periods after an injection of relativistic electrons (cf. V.I. Slysh, 1992, *ApJ*, **391**, 453).

5.4 The Scattered Photon Field

For compact synchrotron sources, the photon energy density due to synchrotron emission may be sufficiently high that the electrons lose a significant portion of their energies to the inverse Compton effect off their own photons. Recall that the "first generation" of IC photons has the power

$$P_{\rm IC} = P_{\rm S}\left(\frac{U_{\rm R}}{U_{\rm M}}\right) = P_{\rm S}\,R \quad \text{where} \quad R \equiv \frac{U_{\rm R}}{U_{\rm M}} \ .$$

Note that once again the symbol 'R' is being used for two purposes.

The quantity $U_{\rm R}$ is the synchrotron photon energy density. It does not include the (higher energy) scattered photons against which the electrons can lose even more energy. That is, the *effective* $U_{\rm R}$ is actually higher, increasing the scattered IC power. This incremental photon density can be expressed as

$$\Delta U_{\rm R} = \left(\frac{P_{\rm IC}}{P_{\rm S}}\right) U_{\rm R} = R\,U_{\rm R} \ .$$

Call this the *first-order* scattered field. Thus, the actual photon energy density is $U_{\rm R} + \Delta U_{\rm R}$ and the modified inverse Compton power becomes

$$P_{\rm IC} = P_{\rm S}\frac{(U_{\rm R} + R\,U_{\rm R})}{U_{\rm M}} = P_{\rm S}(R + R^2) \ .$$

We can repeat this argument recursively because the increase in $P_{\rm IC}$ leads to still another (second-order) scattered photon field which acts as a further target for electrons. We have by analogy then

$$P_{\rm IC} = P_{\rm S}(R + R^2 + R^3 + R^4 + \ldots)\,,$$

an infinite series which clearly diverges if $R > 1$. However, remember that we have assumed *Thomson* scattering in the electron's rest frame. Each higher order scattered field has higher and higher energy due to the Inverse Compton blue-shift. Above some order, the energy will exceed the $\gamma\epsilon \sim mc^2$ limit for Thomson scattering. At this point (and above), the Klein-Nishina cross-section drops off, effectively squelching the IC process. Therefore, our infinite series has only a *finite* number of terms in practice.

The maximum order of the scattered field that is relevant is N defined by

$$\epsilon_{\rm N}^{'} \sim \gamma\epsilon_{\rm N} \sim mc^2$$

where $\epsilon_{\rm N}$ is the mean photon energy associated with the Nth order scattered field. After N scatterings, the photon energy has climbed to

$$\epsilon_{\rm N} \simeq \epsilon\,(\gamma^2)^N$$

and so we cut off the multiple scatterings at the point where

$$\epsilon\,\gamma^{2N+1} \sim mc^2 \ .$$

Above this level, Compton scattering drops off like ϵ^{-1} as we have seen. Assume ϵ is the energy of a *radio* photon (10^8 Hz, say). Then,

$$\epsilon = h\nu \sim 6\times 10^{-27}\times 10^{8} \sim 6\times 10^{-19} \sim 10^{-18}\,\text{ergs} \ .$$

Recall that

$$mc^2 \sim 0.5\,\mathrm{Mev} \sim 0.5 \times 1.6 \times 10^{-6} \sim 10^{-6}\,\mathrm{ergs}\ .$$

Therefore,

$$(2N+1)\log\gamma \sim \log\left(\frac{10^{-6}}{10^{-18}}\right) \sim 12\ .$$

Set $\gamma \sim 10^2$ or $\log\gamma = 2$. We now have

$$2N+1 \sim 6 \qquad \Longrightarrow \qquad N \sim 2.5\ .$$

Hence, only 2–3 generations of scatterings are important before the Klein-Nishina cutoff occurs. But if $R > 1$, this represents a catastrophic drain on the electron energy. The result will be a reduction in the synchrotron power and the corresponding photon energy density. Thus, one should not expect to see an R much greater than *unity* in compact sources. This limiting state corresponds to an equipartition between radiation and magnetic fields.

Computing the evolution of the spectrum in the presence of repeated scatterings off relativistic electrons is a very difficult problem. One has to integrate the time-dependent Boltzmann equation (see text). It is difficult to make any general conclusions other than that the spectrum will still be power-law, much like the original synchrotron spectrum but with its peak shifted to the blue.

For many sources (such as galactic SN remnants), R is fairly small (reflecting the optically thin nature of the source) and so the spectrum is not distorted by IC scattering very much. The particle density is too low for any significant self-absorption; therefore, the radiation field can't build up its energy content. The particles themselves, however, are trapped until their energies become large enough ($U_{\mathrm{P}} \sim U_{\mathrm{M}}$) to permit escape. For some very compact extragalactic sources (e.g., quasars), however, R might approach unity. To estimate the effective R in such cases might require comparing the observed spectrum with models or simulations of the scattered field.

The Sunyaev-Zel'dovich Effect

References:

- Rybicki and Lightman
- Peebles, J.E. *Principle of Physical Cosmology*, p581
- Rephaeli, Y. 1995, *Ann. Rev. Ast. Ap.*, **23**, 541

This effect describes the scattering of the cosmic microwave background radiation by *non-relativistic* but hot gas in clusters. This is analogous to the inverse Compton effect that we have been discussing but the phenomenon is shifted far down in frequency.

The hot electrons in the intracluster medium scatter microwave photons displacing them upward in frequency (because the electrons have more energy than the photons). In particular, they are shifted from below the peak of the blackbody curve to above it. This causes a drop in the apparent microwave background temperature below the peak and a rise on the other side. This is referred to as the "thermal Sunyaev-Zel'dovich effect". It is very small ($\delta T/T \sim 10^{-5} - 10^{-4}$) but in the past few years, improvements in instrumentation have led to positive detections. The interesting aspect of this problem is that it represents an independent way of measuring the Hubble constant as we shall see.

The S.-Z. effect involves multiple scatterings and the change in photon population number n_ν as described by the "Kompaneets equation" (1957 Soviet Physics JETP, **4**, 730). The derivation of the equation is too long to include here but suffice it so say that it accounts for the change in the phase space density of photons due to collisions or scatterings off electrons. If the radiation was in thermodynamic equilibrium with the gas, then of course there would be no effect. But in this case, the microwave photons are *not* in equilibrium with the hot cluster gas and they will try to shift their distribution in energy towards equipartition. The extent to which they succeed will depend on how long the photons on average spend in the cluster; that is, the S.-Z. effect depends on the optical depth of the cluster and, hence, on the cluster's physical dimension which is where H_0 comes in.

What we want is an equation for $\partial n_\nu/\partial t$ in the limit $h\nu \ll kT_e$ and for an isotropic velocity distribution of electrons and photons. This is the Kompaneets equation:

$$\frac{\partial n_\nu}{\partial y} = \nu^2 \frac{\partial^2 n_\nu}{\partial \nu^2} + 4\nu \frac{\partial n_\nu}{\partial \nu} \qquad \text{(Peebles, eqn. 24.47)}$$

where y is sometimes referred to (imaginatively) as the "Compton y-parameter". It is a dimensionless time variable; its incremental value is defined by

$$dy = \frac{kT_e}{mc^2}\, \sigma_{\mathrm{T}} N_e c\, dt \ .$$

If the gas is optically thin with a uniform kinetic temperature, then we can integrate along a photon path and relate y to the photon's mean optical depth:

$$y = \frac{kT_e}{mc^2}\, \sigma_{\mathrm{T}} \int_0^t N_e\, ds = \frac{kT_e}{mc^2}\, \tau \ .$$

We can now solve for the incremental change in n_ν over the optical path τ by substituting the expression for the blackbody occupation number $n_\nu = (\mathrm{e}^x - 1)^{-1}$ where $x \equiv h\nu/kT_{\mathrm{CMB}}$:

$$\frac{\delta n_\nu}{n_\nu} = y \left[\frac{x^2 \mathrm{e}^x (\mathrm{e}^x + 1)}{(\mathrm{e}^x - 1)^2} - \frac{4x\mathrm{e}^x}{(\mathrm{e}^x - 1)} \right]$$

which asymptotically approaches $-2y$ for small x, the Rayleigh-Jeans regime.

Now from $\delta n_\nu/n_\nu$, we can estimate the relative change in the radiation (or brightness) temperature:

$$\delta n_\nu = -\frac{\delta x}{(\mathrm{e}^x - 1)^2} \qquad \text{where} \qquad \delta x = -\frac{h\nu}{kT^2}\, \delta T = -x \frac{\delta T}{T} \ .$$

Thus,

$$\frac{\delta n_\nu}{n_\nu} = \frac{(\mathrm{e}^x - 1)}{(\mathrm{e}^x - 1)^2}\, x\, \frac{\delta T}{T} \sim \frac{\delta T}{T} \qquad (x \ll 1) \ .$$

Therefore, the magnitude of the thermal S.-Z. effect is given by

$$\frac{\delta n_\nu}{n_\nu} = \frac{\delta T}{T} = -2y \ .$$

This is the basic equation for the drop in brightness temperature, now observable by radio telescopes at millimeter wavelengths which lie just *above* (in wavelength) the microwave peak at 1.38 mm. It was first derived by Sunyaev and Zel'dovich in 1972 (Comm. Astrophys. Space Phys., **4**, 173).

To evaluate the order of magnitude of the S.-Z. effect in °K, we adopt the following simple model for the intracluster plasma distribution:

$$N(r) = \frac{N_c}{(1 + r^2/r_c^2)} \qquad (r_c \equiv \text{core radius})\,.$$

Then, the optical depth through the cluster center is

$$\tau_o \equiv \sigma_{\mathrm{T}} \int_{-\infty}^{\infty} N(r)\, dr = 0.0064 \left(\frac{N_c}{\mathrm{cm}^{-3}}\right) \left(\frac{r_c}{\mathrm{kpc}}\right) \,.$$

Note that the S.-Z. effect does not depend on knowing the distance to the cluster. But rather the measurement of $\delta T/T$ will give directly the physical extent of the cluster (r_c) and, thereby, its distance which in turn yields a value for the Hubble constant H_0.

We can infer information concerning N_c and r_c from the X-ray emission (bremsstrahlung) of the hot intracluster gas. The expression for the emissivity on page 50 (§3.1) can be used; setting $Z = 1$ and ignoring the Gaunt factor, we have

$$\epsilon_{\mathrm{ff}} \simeq 1.4 \times 10^{-27}\, T^{1/2}\, N^2 \ \mathrm{ergs\,s^{-1}\,cm^{-3}} \,.$$

The X-ray luminosity then becomes

$$L_x = 4\pi \int_0^{r_c} \epsilon_{\mathrm{ff}}\, r^2\, dr = 2.5 \times 10^{41}\, f^{-1} \left(\frac{N_c}{\mathrm{cm}^{-3}}\right)^2 \left(\frac{r_c}{\mathrm{kpc}}\right)^3 \left(\frac{kT_x}{\mathrm{kev}}\right)^{1/2} \ \mathrm{erg\,s^{-1}} \,,$$

where $f \equiv V_x/V_c$ is a filling factor equal to the ratio of the X-ray emitting volume and the cluster core volume. Note that this expression allows for the variation of $N(r)$ through the core. From observation, we have typically

$$\begin{aligned} L_x &\sim 10^{44}\, h^{-2}\, \mathrm{erg\,s^{-1}} \quad (\mathrm{h} \equiv H_0/100\,\mathrm{km\,s^{-1}\,Mpc^{-1}})\,, \\ r_c &\sim 200\, h^{-1}\, \mathrm{kpc} \quad (\text{from angular diameter})\,, \\ T_x &\sim 4\,\mathrm{kev}\,. \end{aligned}$$

Thus, from the luminosity equation, we have

$$N_c \sim 0.0050\, h^{1/2}\, f^{-1/2}\, \mathrm{cm}^{-3}$$

and, from the equation for τ_o, we find

$$\tau_o = (0.0050\, h^{1/2} f^{-1/2})(200\, h^{-1})(0.0064) = 0.0064\, (hf)^{-1/2} \,.$$

Finally, the temperature decrement of the microwave radiation passing through the cluster is

$$\frac{\delta T}{T} = -2\tau_o \left(\frac{kT_x}{mc^2}\right) = -1.0 \times 10^{-4}\, (f h^{-1})^{1/2} \,.$$

This is small but now within the realm of observation and can (potentially) yield a value for the Hubble parameter h (or H_0) although it would be somewhat dependent on the modeling of the cluster density distribution including the volume filling factor of the X-ray emission. In Peebles' words, this requires an "heroic mapping" of the X-ray emission and the temperature in the cluster. All we have done here is make an order of magnitude estimate of $\delta T/T$. The people who actually try to determine h from the temperature deficit will have to be much more careful about the modeling.

The observational situation as of 2001 is

Cluster	Reference	H_0 (km s^{-1} Mpc^{-1})
A2218	Birkinshaw (1990)	$50 \pm$?
A2218	McHardy *et al.* (1990)	24 ± 12
A665	Birkinshaw (1991)	45 ± 12
Coma	Myers *et al.* (1994)	74 ± 29
A2142	"	56 ± 11
A2256	"	76 ± 20
Set of 8	Hughes & Birkinshaw (1998)	53 ± 17
Set of 7	Mason *et al.* (2001)	66 ± 15
Mean (post 1992)		61 ± 17

where the sources are

Birkinshaw	*The Cosmic Microwave Background: 25 Years Later*, (eds. N. Mandolesi & N. Vittorio) 1990
McHardy *et al.*	M.N.R.A.S., **242**, 215, 1990
Birkinshaw	*Ap.J.*, **379**, 466, 1991
Myers *et al.*	preprint (1994)
Hughes & Birkinshaw	*ApJ*, **501**, 1, 1998
Mason *et al.*	*ApJ*, **555**, L11, 2001

See P. R. Phillips, (*Ap.J.*, **455**, 419, 1995) for a derivation of the "kinetic" Sunyaev-Zel'dovich effect from the Boltzmann equation. This effect arises from the fact that the cluster as a whole has a drift velocity relative to the CMB along the line of sight. This introduces an overall Doppler shift of the spectrum which by itself can introduce a $\delta T/T$ at a particular wavelength. However, such drift velocities are probably small which makes the kinetic effect less important (we expect) than the thermal one.

See the next page for a sample set of observations which shows the differential sky brightness at $\lambda = 2.2\,\mathrm{mm}$ across the cluster A2163. A 3×3 array of bolometers is used to accumulate the data. By taking appropriate differences, one can eliminate the effects of background.

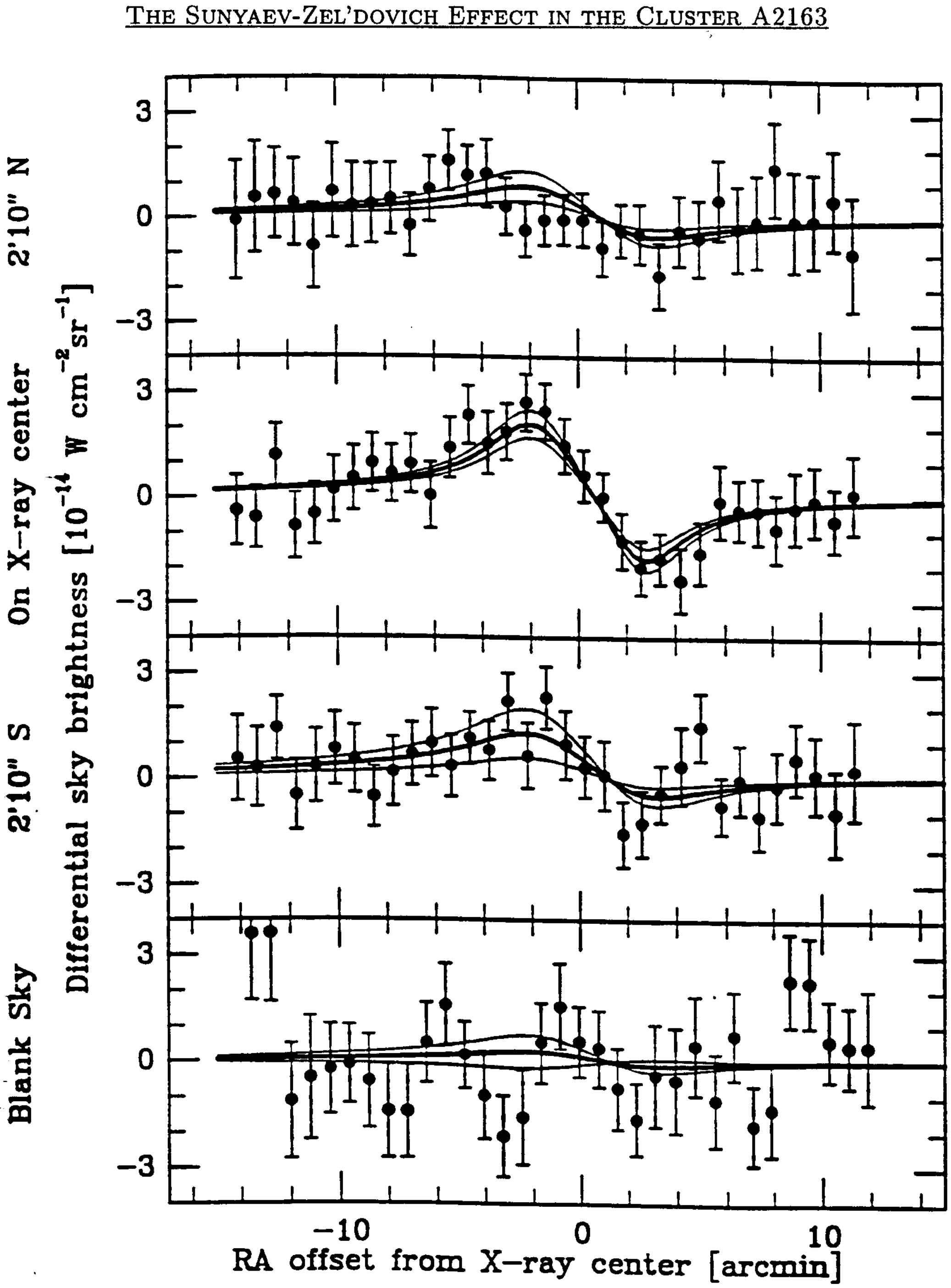


Figure 5 – Difference signals from scans of A2163 at three declinations plus the blank sky. The four panels correspond to the 4′.3 difference on A2163 at the 14′.5 RAO for April 25 (top) and April 26 (upper and lower middle), and to the 4′.3 difference on blank sky at the 14′.0 RAO on April 26. The data shown constitute 3/36 of the total data in all 12 sets on A2163. The best fit signal template for each of the data sets is shown by the heavy curve, and has been fit to all three differences. The amplitudes of the $\pm 1\sigma$ uncertainties are indicated by the light curves. The vertical axis is the brightness of a source that fills one beam completely and fills none of the other beam. (After Wilbanks et al. 1994, *ApJ Letters*, **427**, L75.)