4. Synchrotron Radiation (Magneto-Bremsstrahlung)

4.1 Emission by a Single Electron in a Vacuum

Preamble:

It is useful to keep in mind that ordinary electrons at $10^4\,°$K (H II regions, for example) are much stronger sources of thermal (radio) emission than synchrotron radiation. This is because the characteristic thermal bremsstrahlung frequency is $v/b \sim 10^8\,\mathrm{cm\,s^{-1}}/0.1\,\mathrm{cm} \sim 10^9$ Hz whereas the magneto-bremsstrahlung frequency is the cyclotron frequency $\omega_{\mathrm{B}} \sim eB/mc \sim 1$ Hz for a typical galactic field of a few micro-gauss. For this reason, the electrons must be ultra-relativistic to produce a similar power in the radio. We shall see that the basic frequency ω_{B} can be Doppler boosted by a potentially large factor of γ^2.

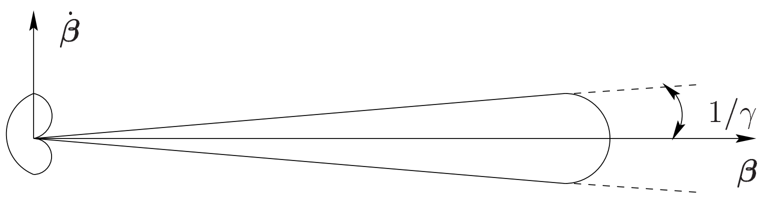
Thus, we are concerned here with radiation by very energetic electrons accelerated by generally weak magnetic fields. They are typically cosmic-ray electrons. The phenomenon is the high energy analog of the cyclotron radiation emitted by nonrelativistic electrons ($\gamma = 1$). As demonstrated in the first paragraph, the latter is not normally observable in astrophysical situations due to very low magnetic field strengths.

In situations where particle velocities are relativistic, the densities are generally *low* and so synchrotron emission tends to dominate thermal bremsstrahlung since the emissivity of the latter goes like the *square* of the density.

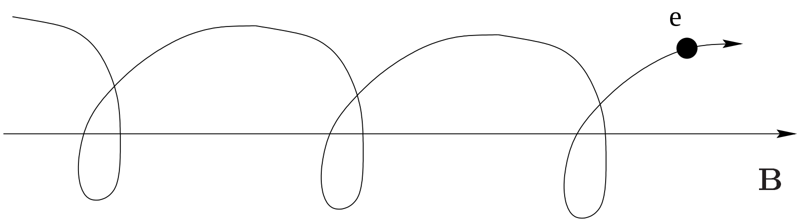
We saw earlier that the $\boldsymbol{E}$-fields of accelerated relativistic particles can become very large in the direction of motion; i.e., when

$$1 - \boldsymbol{n}\cdot\boldsymbol{\beta} \longrightarrow 0 \qquad (|\boldsymbol{\beta}| \to 1)\,.$$

In this case, the radiation is beamed into an angle $\theta \sim 1/\gamma = mc^2/E$:



This alters the character of the radiation significantly; that is, there are very sharp pulses which means that there are high frequency components in the signal. The electron, of course, follows a *helical* path about an axis coincident with a magnetic field line:



The gyromagnetic frequency is ω_{B}. Since the force here, $\frac{e}{c}\,(\boldsymbol{v} \times \boldsymbol{B})$, is perpendicular to $\boldsymbol{v}$, the speed

v does not change and so γ remains constant. Therefore,

$$\frac{d}{dt}(\gamma m \boldsymbol{v}) = \gamma m \frac{d\boldsymbol{v}}{dt} = \frac{e}{c}(\boldsymbol{v} \times \boldsymbol{B}) \ .$$

In the appropriate frame of reference, the orbit is circular:

$$\frac{dv_{\|}}{dt} = 0 \ , \qquad \frac{dv_{\perp}}{dt} = \omega_{\rm B}^2 r \qquad \text{(harmonic motion)}$$

or

$$\gamma m \omega_{\rm B}^2 r = \frac{e}{c}\,\omega_{\rm B} r B \qquad \Longrightarrow \qquad \omega_{\rm B} = \frac{eB}{\gamma m c} \ .$$

For electrons, this works out numerically to be

$$\omega_{\rm B} = 1.76 \times 10^7 \,\frac{B}{\gamma} \qquad \left(\gamma = \frac{E}{mc^2}\right) \ .$$

In the interstellar medium, $B \sim 10^{-6}$ to 10^{-5} gauss. Take 3×10^{-6} gauss at a nominal value and $\gamma = 1$; then,

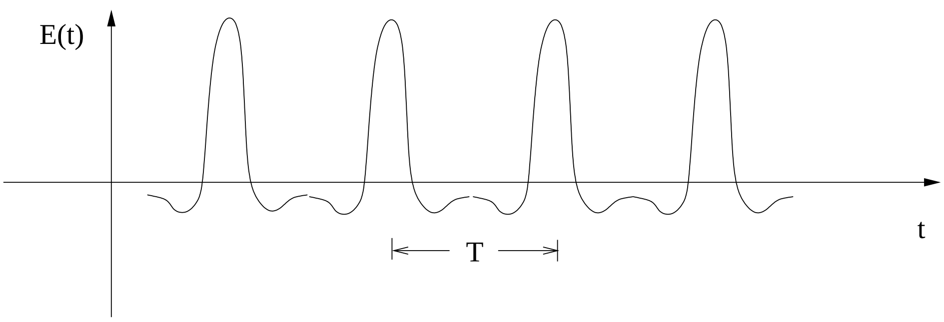
$$\omega_{\rm B} \sim 10^7 \times 3 \times 10^{-6} \sim 30\,\text{Hz} \qquad \text{(nonrelativistic)} \, .$$

For cosmic-ray electrons, $\gamma \sim 10^3$ and, hence, $\omega_{\rm B}$ is significantly less than 1 Hz because the radius of its helical orbit is very large.

A complete mathematical treatment of synchrotron radiation is offered in a number of references, including:

- Legg, M.P.C. & Westfold, K.C. 1968, *Ap.J.*, **154**, 499
- Pacholczyk, A., *Radio Astrophysics*
- Ginzburg & Syrovatskii 1965, *Ann. Rev.*, **3**, 297
- ________ 1969, *Ann. Rev*, **7**, 375
- Rybicki & Lightman, *Radiative Processes*

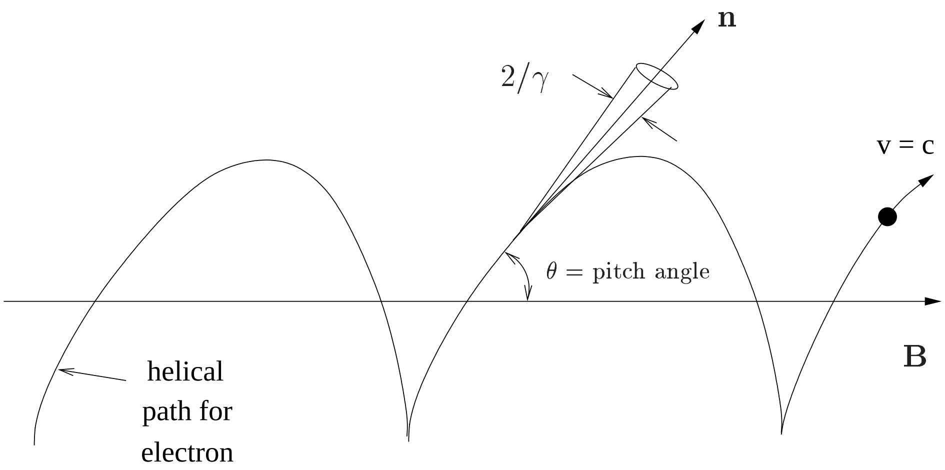
In this course, I want to stress a qualitative treatment which yields more physical insight. The quantitative treatment (in the above references) starts with the expressions for $\boldsymbol{E}(\boldsymbol{n}, \boldsymbol{\beta}, \dot{\boldsymbol{\beta}})$ and $\boldsymbol{B} = \boldsymbol{n} \times \boldsymbol{E}$ and shows that the synchrotron motion leads to a series of sharply defined pulses, repeating at the gyromagnetic frequency; more specifically, the observed pulse period is $T = 2\pi \sin^2\theta / \omega_{\rm B}$ as shown on the next page.



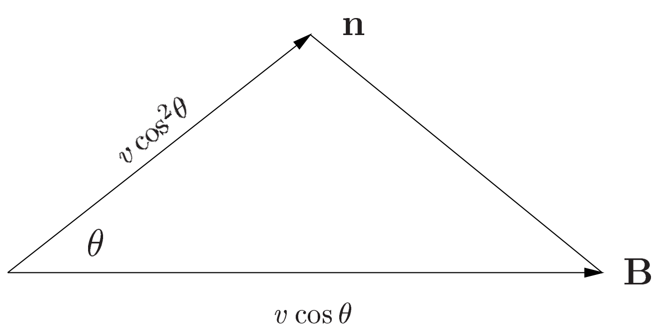
Let's begin with a qualitative examination of these pulses. Keep in mind that the departure of $E(t)$ from a simple harmonic variation (viz., $\sin \omega t$) is due to the combination of Doppler shifts (the $1 - \boldsymbol{n} \cdot \boldsymbol{\beta}$ term) and the fact that in the vector potential $\boldsymbol{A}$ and the scalar potential ϕ the retarded times are *different* at different parts of the electron's orbit. This makes the effective charge distribution (responsible for the potentials) change from that of a simple rotating dipole to instead a superposition of a dipole ($\Rightarrow \omega_{\mathrm{B}}$), a quadrapole ($\Rightarrow 2\omega_{\mathrm{B}}$), a sextapole ($\Rightarrow 3\omega_{\mathrm{B}}$), an octopole ($\Rightarrow 4\omega_{\mathrm{B}}$), etc. Thus, the Fourier transform of $E(t)$ contains *discrete* components; i.e., it is NOT a continuum! (See §4.4 for more details.)

Qualitative Treatment:

(a) Spectrum



The frequency of gyration is $\omega_{\mathrm{B}} = eB/\gamma mc$. If the orbit were purely circular (i.e., $\theta = \pi/2$), then the observer would detect a series of pulses with $P = 2\pi/\omega_{\mathrm{B}}$. However, since the electron's guiding center is moving with the velocity $v\cos\theta$ along the field line, and since this motion has a component projected toward the observer of $v\cos^2\theta$, there is a "Doppler compression" of the pulse period.



The pulses are spaced apart by a distance Δs:

$$\Delta s = Pc - v\cos^2\theta\, P = Pc\left(1 - \frac{v}{c}\cos^2\theta\right) .$$

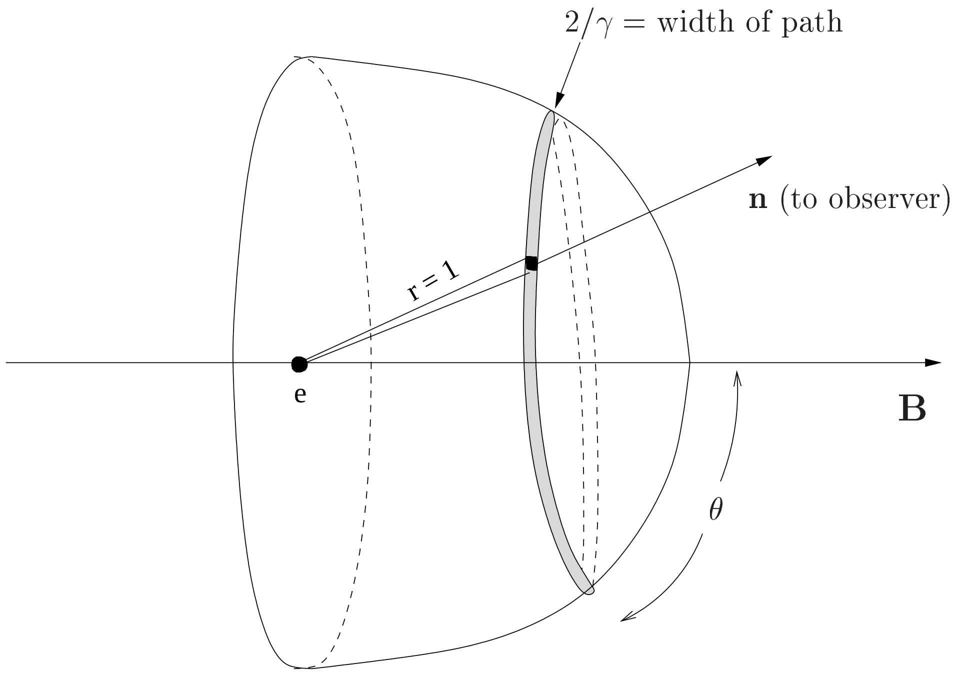
Since $v \sim c$,

$$\Delta s = Pc\,(1 - \cos^2\theta) = Pc\,\sin^2\theta\ .$$

Hence, the observed period is

$$T = \frac{\Delta s}{c} = P\,\sin^2\theta\ .$$

The *pulse width* is also subject to a Doppler compression:



The shaded strip here is the path on the celestial sphere swept out by the radiating cone of the electron. The width of the pulse, $\Delta t'$, is determined by the fraction of the gyromagnetic period P that the electron is radiating toward the observer (i.e., the black circle on the path):

$$\frac{\Delta t'}{P} = \frac{2}{2\pi\gamma\sin\theta} \quad \text{or} \quad \Delta t' = \frac{P}{\pi\gamma\sin\theta} = \frac{2}{\gamma\omega_{\mathrm{B}}\sin\theta} = \frac{2mc}{eB\sin\theta}$$

which is *independent* of γ! This pulse is subject to a Doppler compression since the particle is instantaneously moving directly toward the observer with velocity v:

$$\Delta t = \Delta t'\left(1 - \frac{v}{c}\right)\ .$$

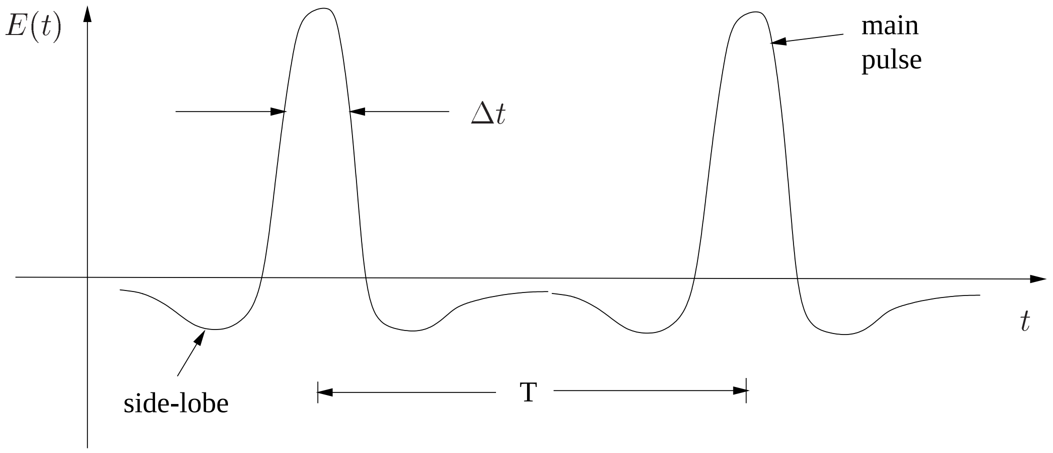
But,

$$\frac{1}{\gamma^2} = \left(1 - \frac{v^2}{c^2}\right) \simeq 2\left(1 - \frac{v}{c}\right)\ .$$

Therefore, the compression factor is $1 - v/c \sim 1/(2\gamma^2)$. Now, we can write

$$\Delta t = \frac{1}{\pi\gamma\sin\theta} \times \frac{2\pi}{\omega_{\mathrm{B}}} \times \frac{1}{2\gamma^2} = \frac{1}{\gamma^3\,\omega_{\mathrm{B}}\,\sin\theta}\ .$$

Putting this altogether, we can make a sketch of $E(t)$:



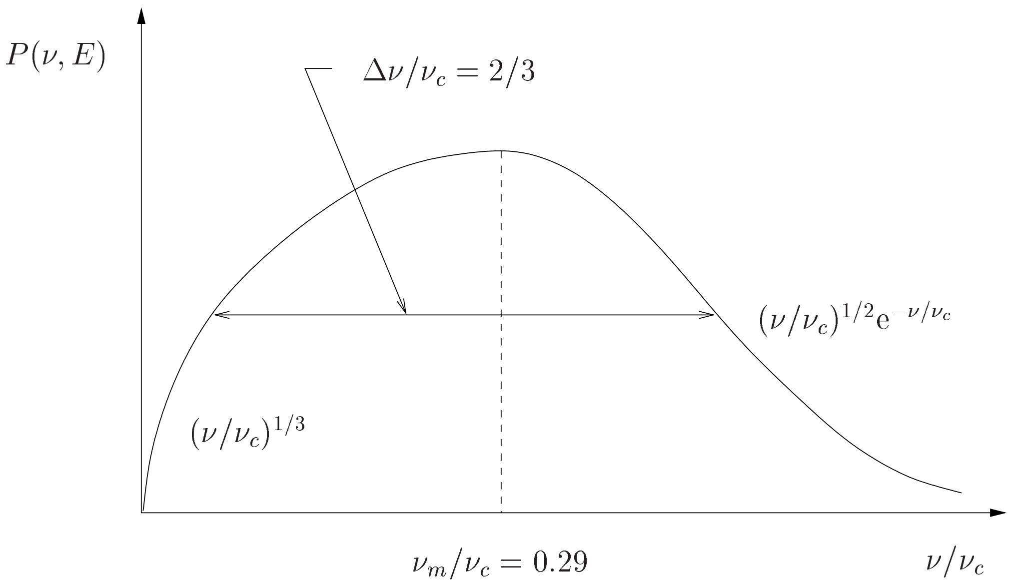
This is obviously not to scale because

$$\Delta t = \frac{1}{\gamma^3 \omega_{\mathrm{B}} \sin\theta} \ll T = \frac{2\pi}{\omega_{\mathrm{B}}} \sin^2\theta$$

when γ is large.

To get the spectrum, we simply take the Fourier transform of this pulse train. This results in a fairly broad spectrum because the pulses consist of many *high* frequency components.

$$\text{bandwidth} = \Delta\omega \sim \Delta t^{-1} = \gamma^3\, \omega_{\mathrm{B}}\, \sin\theta\ .$$



This spectrum is actually *discrete* (see §4.4). The quantity ν_c is called the "critical frequency" and is given by

$$\nu_c \equiv \frac{3\gamma^3}{4\pi}\, \omega_{\mathrm{B}}\, \sin\theta = \frac{3}{2}\, \Delta\nu\ .$$

The spectrum peak occurs at

$$\nu_m = 0.29\,\nu_c = 1.26 \times 10^6\, B\gamma^2 \ .$$

These frequencies (ν_c and ν_m) are of order

$$\frac{30\,\text{Hz}}{\gamma} \times \gamma^3 \simeq 3 \times 10^9\,\text{Hz} \quad (\text{for } \gamma \simeq 10^4) \ .$$

This frequency falls into the *observable radio spectrum* of cosmic-ray electrons. That is, the relativistic motion has boosted the frequency up by a factor of 10^8. Or, another way of putting it, we are seeing a very high harmonic of the gyrofrequency ω_{B}:

$$n \sim \frac{\omega_c}{\omega_{\text{B}}} \simeq \gamma^3 \qquad \Longrightarrow \qquad 10^{12}\text{ th harmonic for } \gamma = 10^4 \ !!$$

(b) Power Radiated by a Single Electron

We could integrate $P(\nu, E)$ over all ν but it is easier to invoke the relativistic Larmor formula:

$$P = \frac{2e^2}{3c^3}\,\gamma^4 (a_\perp^2 + \gamma^2 a_{||}^2)$$

where $a_\perp$ and $a_{||}$ are the acceleration components perpendicular and parallel to the electron's motion. In this case, $a_{||} = 0$ and

$$a_\perp = \frac{evB\sin\theta}{\gamma mc} \sim \frac{eB\sin\theta}{\gamma m} \qquad (v \sim c) \ .$$

Therefore,

$$P(E) = \frac{2e^2}{3c^3}\,\gamma^4 \left(\frac{eB\sin\theta}{\gamma m}\right)^2 = \frac{2e^4}{3m^2c^3}\,B^2\gamma^2\sin^2\!\theta \ \propto \ E^2 \ .$$

Note that both the power and the critical frequency ν_c depend on the product $B\sin\theta \equiv B_\perp$, the field component normal to the electron's velocity vector. This is the component which is responsible for accelerating the electron (viz., $|\boldsymbol{a}| \propto |\boldsymbol{v} \times \boldsymbol{B}| = vB\sin\theta$).

Suppose we had an isotropic velocity distribution or a random field orientation. Then the *average power per electron* could be found by averaging over the pitch angle. The probability of a single electron having a pitch angle between θ and $\theta + d\theta$ is

$$\frac{2\pi\sin\theta\,d\theta}{4\pi} = \frac{1}{2}\,\sin\theta\,d\theta \ .$$

Therefore,

$$\begin{aligned}
\bar{P}(\gamma) &= \frac{2e^4B^2\gamma^2}{3m^2c^3}\int_0^\pi \sin^2\!\theta\,\frac{1}{2}\sin\theta\,d\theta = \frac{4e^4B^2\gamma^2}{9m^2c^3} \\
&= \frac{4}{3} \times \frac{8\pi e^4}{3m^2c^4} \times c \times \frac{B^2}{8\pi} \times \gamma^2 \\
&= \frac{4}{3}\,\sigma_{\text{T}}\,c\,U_{\text{B}}\,\gamma^2
\end{aligned}$$

In the limit $\gamma \to 1$, we should replace γ^2 in the foregoing equation by $\gamma^2 - 1 = \gamma^2\beta^2 \propto v^2$. Thus, in the non-relativistic limit, $P(E) \propto E$ as opposed to E^2.

This expression for $\bar{P}$ lends itself to an interesting physical interpretation. As seen by an electron, the magnetic energy flowing past it is cU_{B} per unit area. If we multiply this by the scattering cross-section σ_{T} per electron and apply a "boost" factor γ^2 to the "scattered energy", we get the synchrotron power!! Thus, synchrotron emission can be viewed as the scattering of magnetic energy by relativistic electrons. But this is a distorted view because the magnetic field is not the source of the synchrotron power; rather, it is the kinetic energy of the electrons.

(c) Synchrotron Loss-time

Given the energy loss rate, we can easily find the half-life of a synchrotron emitting electron.

$$\frac{dE}{dt} = -P(E) = -\beta E^2$$

where

$$\beta = \frac{2e^4 B^2 \sin^2\theta}{3m^4 c^7} \ .$$

Solving, we find

$$E(t) = \frac{E_o}{(1 + \beta E_o\, t)} \ ,$$

where $E_o \equiv E(0)$. The half-life is clearly

$$t_{1/2} = \frac{1}{\beta E_o} = \frac{5.1 \times 10^8}{B_\perp^2} \left(\frac{mc^2}{E_o} \right) \mathrm{s} = \frac{5.1 \times 10^8}{B_\perp^2 \gamma_o} \mathrm{s} \ .$$

Example: $\gamma_o = 10^3, \quad B = 10^{-5}$ gauss:

$$t_{1/2} = \frac{5.1 \times 10^8}{(10^{-5})^2} \times \frac{1}{10^3} \ \simeq \ 5 \times 10^{15}\,\mathrm{s} \ \simeq \ 10^8\,\mathrm{yr} \ .$$

4.2 Volume Emissivity of a Power-Law Distribution

In a wide range of astrophysical applications, the energy spectrum of relativistic electrons is *power-law* as might be produced by a stochastic acceleration mechanism. A good example is the Fermi mechanism which operates in supernova remnants: electrons scatter off turbulent magnetic "bubbles" and are pushed towards equipartition but before they can achieve statistical equilibrium they escape the remnant around energies of 10^{15} ev. The resulting energy distribution looks like

$$N(E)\,dE = \begin{cases} KE^{-p}\,dE & (E_1 \le E \le E_2)\ , \\ 0 & (\text{otherwise})\ . \end{cases}$$

Here, p is called the energy spectral index (~ 2.5 for cosmic rays). To compute the emissivity or the emission coefficient, we assume (i) uniform magnetic field, (ii) power-law energy distribution, and (iii) isotropic velocities.

Given the frequency spectrum for electrons of a given energy, we can write

$$j_\nu = \frac{1}{4\pi}\int_0^\infty P(\nu, E)\, N(E)\, dE\ .$$

This integral is tough to evaluate accurately, but we can get a good approximation by assuming that all the electrons radiate at their critical frequency, ν_c. Then, per unit solid angle,

$$j_\nu\, d\nu = \frac{N(E)\,dE}{4\pi} \times P(E) = \frac{KE^{-p}}{4\pi} \times \frac{2e^4 B_\perp^2}{3m^2c^3}\left(\frac{E}{mc^2}\right)^2 dE\ .$$

Now substitute for E and dE in terms of ν and $d\nu$:

$$\nu = \nu_c = \frac{3}{4\pi}\,\omega_{\mathrm{B}}\sin\theta\left(\frac{E}{mc^2}\right)^3 = \frac{3}{4\pi}\left(\frac{eB_\perp}{\gamma mc}\right)\left(\frac{E}{mc^2}\right)^3\ .$$

After some reduction, one finds

$$j_\nu = \frac{1}{9}\left(\frac{e^3}{mc^2}\right)\left(\frac{3e}{4\pi m^3c^5}\right)^{(p-1)/2} KB_\perp^{(p+1)/2}\,\nu^{(1-p)/2}\ .$$

This formula is, of course, approximate but it differs from the exact expression only by a numerical factor of order unity. In particular, it has the correct spectral index $\alpha = (1-p)/2$. For cosmic-ray electrons, $p \simeq 2.5$ which implies that $\alpha \simeq -0.75$. Thus, j_ν falls off relatively slowly with frequency compared with thermal radiation. Note that the formula is not consistent with an isotropic magnetic field distribution. In the latter case, we replace $B_\perp$ with $B\sin\theta$ and average over all θ. But this would change only the coefficient and not the dependences on B and ν.

A more precise (but still approximate) formula for j_ν can be found on p95 of Pacholczyk's *Radio Astrophysics*:

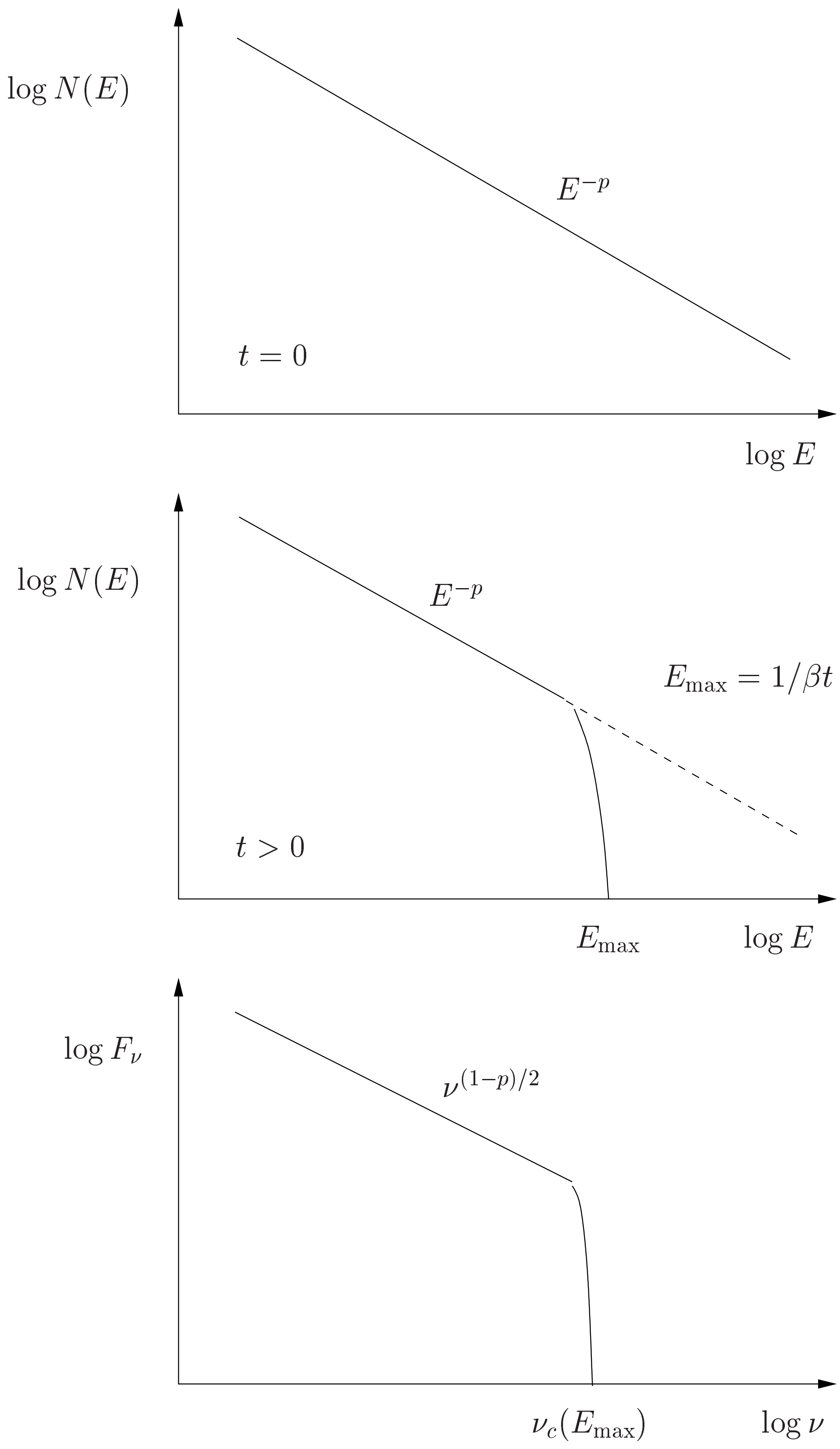
$$j_\nu = C_5(p)\,KB_\perp^{(p+1)/2}\left(\frac{\nu}{2C_1}\right)^{(1-p)/2}$$

where C_1 and C_5 are tabulated in an appendix to the book. This version is based on the correct integration over energy but again it has not been averaged over θ. See page 70 of these notes for the exact formula for j_ν.

Note that radiation losses by the high energy particles will lead to an abrupt cutoff in the spectrum no matter how high the upper limit E_2 initially is. Recall that

$$E = \frac{E_o}{(1+\beta E_o t)} \quad\longrightarrow\quad \frac{1}{\beta t} \quad \left(t \gg \frac{1}{\beta E_o}\right)\ .$$

Thus, after time t, there will be *no* particles with energies above $1/\beta t$!



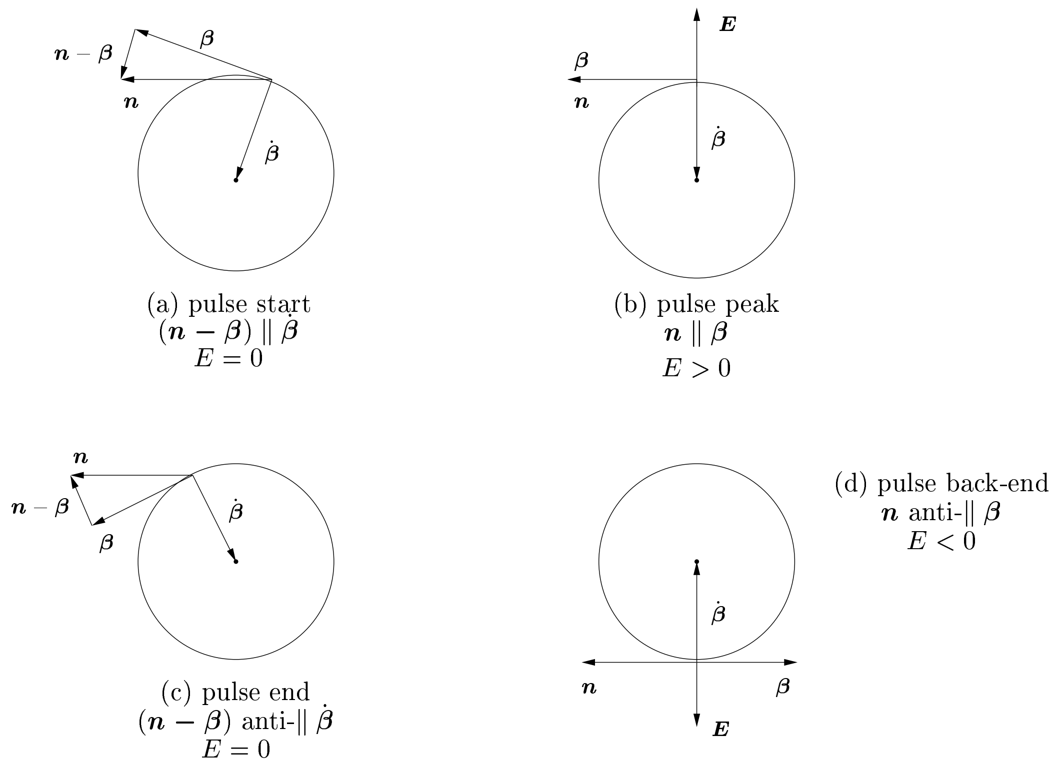
$\log N(E)$
E^{-p}
$t = 0$
$\log E$
$\log N(E)$
E^{-p}
$E_{\max} = 1/\beta t$
$t > 0$
$E_{\max}$
$\log E$
$\log F_\nu$
$\nu^{(1-p)/2}$
$\nu_c(E_{\max})$
$\log \nu$

4.3 Polarization of Synchrotron Radiation

We can describe this qualitatively by recalling that

$$\boldsymbol{E}(\boldsymbol{r},t) \quad \propto \quad \boldsymbol{n} \times \left[(\boldsymbol{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right]$$

and remembering that $\dot{\boldsymbol{\beta}}$ is normal to $\boldsymbol{\beta}$. Consider the following sketch and assume that the observer lies on the track of the beam center:



(a) pulse start
$(\boldsymbol{n}-\boldsymbol{\beta}) \parallel \dot{\boldsymbol{\beta}}$
$E = 0$

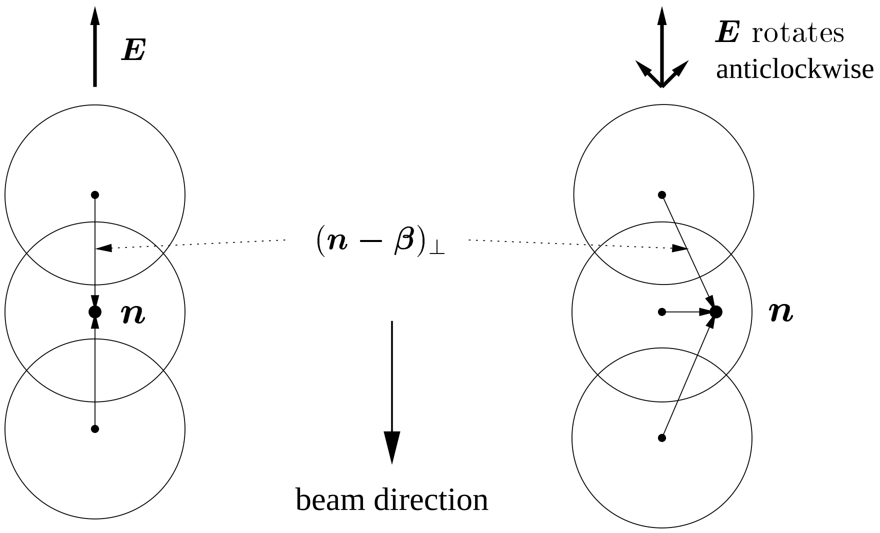
(b) pulse peak
$\boldsymbol{n} \parallel \boldsymbol{\beta}$
$E > 0$

(c) pulse end
$(\boldsymbol{n}-\boldsymbol{\beta})$ anti-$\parallel$ $\dot{\boldsymbol{\beta}}$
$E = 0$

(d) pulse back-end
$\boldsymbol{n}$ anti-$\parallel$ $\boldsymbol{\beta}$
$E < 0$

As the beam sweeps by the observer, $E(t)$ starts at zero, rises to a maximum, and then falls back to zero. The vector $\boldsymbol{E}$ points in the same direction with respect to $\boldsymbol{n}$ throughout the main pulse. Outside the main pulse (i.e., in the side-lobes), $\boldsymbol{E}$ points in the opposite direction and, of course, has a much smaller amplitude. This explains the pulse shape shown on an earlier page.

In any event, it is clear that the beam polarization is basically linear with an orientation normal to the *projected magnetic field* (which lies normal to $\boldsymbol{n}$ and in the plane defined by $\boldsymbol{B}$ and $\boldsymbol{n}$). However, the $\boldsymbol{E}$-vector does *rotate* slightly as it passes by the observer if she is slightly off-center from the pulse beam. The sense of the rotation depends on which side of the beam path the observer finds herself. This introduces a small element of elliptical polarization with an axial ratio $\tan\beta \sim mc^2/E \sim 1/\gamma$ near the edge of the beam. This is represented schematically in the following sketch:



(a) pulse-center
$(\boldsymbol{n}-\boldsymbol{\beta})_\perp$ in fixed plane
linear polarization

(a) off-center
$(\boldsymbol{n}-\boldsymbol{\beta})_\perp$ rotates
partial circular polarization

Thus, the Stokes parameter V which measures circular polarization is small but nonzero:

$$V \propto \sin 2\beta \;\leq\; \frac{1}{\gamma} \sim \left(\frac{\nu_{\mathrm{B}}}{\nu_c}\right)^{1/2} ,$$

where (remember) $\nu_{\mathrm{B}} \equiv eB/2\pi mc$. However, with an ensemble of electrons present, the circular element will almost cancel out because the observer will be on one side of the beam path for half the electrons and on the other side for the remaining half. This leaves a relatively high degree of linear polarization at least for high γ.

Thus, the $\boldsymbol{E}$-vector in synchrotron emission tends to be confined to the plane normal to the projected $\boldsymbol{B}$-field. Therefore, polarization observations provide useful information on field orientations. If the degree of polarization is very small or zero, then the field must have a tangled or random geometry. If it is high, on the other hand, the field must be more or less aligned or uniform. For a power-law distribution of electron energies and random pitch angles, one can show that

$$\Pi = \frac{p+1}{p+\frac{7}{3}} \quad\longrightarrow\quad 72\% \quad (\text{for } p = 2.5) \ .$$

This is the result of averaging over energy (or frequency). In the average over E, the contribution of low energy electrons is negligible and so one can actually integrate over all energies even though $N(E)$ formally blows up for a power-law distribution. Ginzburg and Syrovatskii compute $\Pi(\nu)$ and show that $\Pi \to 1$ at high ν (as the foregoing analysis implies) whereas it goes to $\frac{1}{2}$ at low ν. The latter limit implies that the low frequency radiation from each electron is also elliptically polarized with a much larger axial ratio, namely

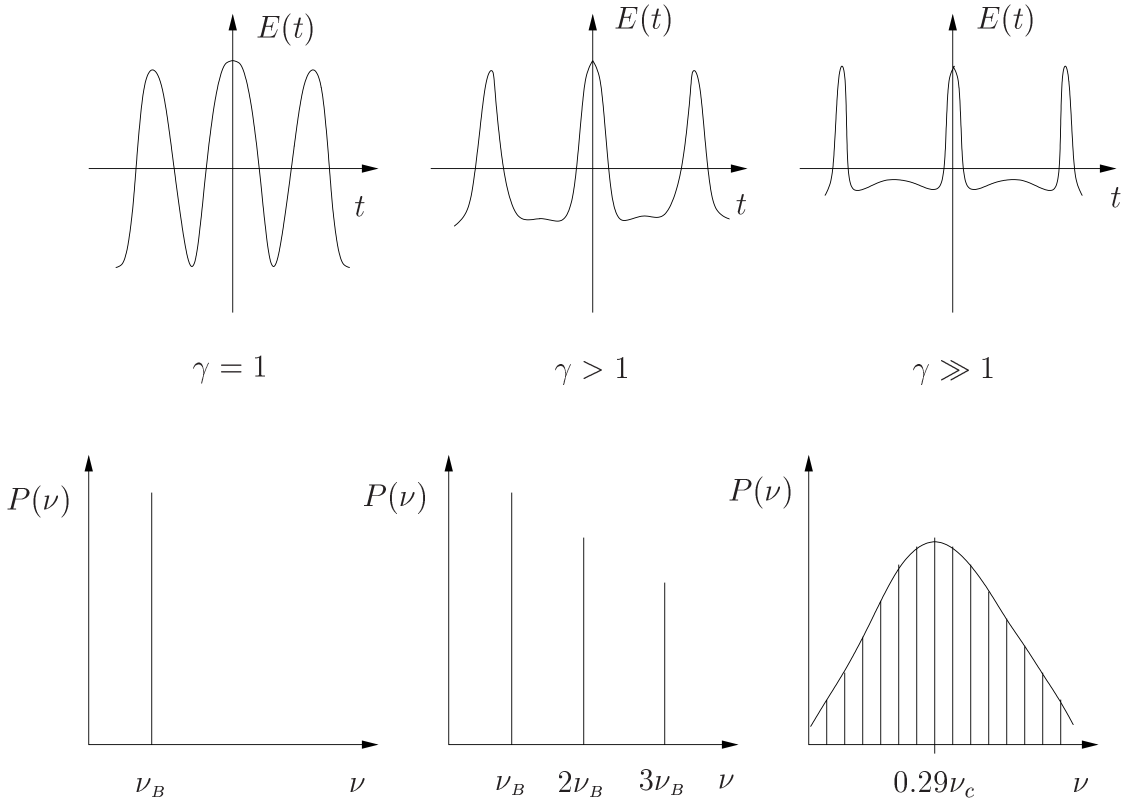
$$\tan\beta = \frac{1}{\sqrt{3}} \quad\Longrightarrow\quad \frac{I_{\min}}{I_{\max}} = \frac{1}{3} \quad\Longrightarrow\quad \Pi = \frac{I_{\max}-I_{\min}}{I_{\max}-I_{\min}} = \frac{1}{2} \ .$$

But the *average* over frequency is 72%. In reality, of course, there will be a cut-off in $N(E)$ at low energies but this should not affect the result. The fact that we see a maximum polarization of about 70% over small regions of supernova remnants confirms the practice of neglecting the effect of low energy electrons.

4.4 Transition From Cyclotron to Synchrotron Emission

The spectrum of a single, circulating (relativistic) charge is not continuous but discrete as explained earlier. That is, such a charge produces radiation at harmonics of the fundamental and, for $v/c \ll 1$, the power of the n^{th} harmonic is proportional to $(v/c)^n$. One can view the effective charge distribution as a multi-pole expansion (or superposition) which arises from the changing retardation times over the electron's orbit and the ever-present Doppler shifts, both of which become more and more pronounced as the particle's energy increases.

When $\gamma = 1$, $E(t)$ is just a sine wave. As γ increases with increasing energy, the $\boldsymbol{E}$-field becomes distorted (due to the Doppler and retardation effects) and harmonics appear. Adopt the convention that $E(t)$ is *positive* at its maximum value; then, schematically the situation looks like the following:



For a distribution of electron energies and perhaps variations in the $\boldsymbol{B}$-field (strength and orientation), the spectrum becomes continuous. That is, one never observes the discrete components.

4.5 Synchrotron Self-Absorption

At low frequencies, synchrotron radiation can be *absorbed* by the very electrons responsible for the emission. This causes the spectrum to turn over and fall to zero as $\nu \to 0$. Another process that can occur in these circumstances is *stimulated* emission. Now that we know the emission coefficient, we can derive the synchrotron absorption coefficient (corrected for stimulated emission) by using the tools already developed (e.g., the Einstein coefficients).

We have seen that

$$\begin{aligned} j_\nu &= \frac{h\nu}{4\pi}\, n_2\, A_{21}\, \phi(\nu) \ , \\ \alpha_\nu &= \frac{h\nu}{4\pi}\, (n_1 B_{12} - n_2 B_{21})\, \phi(\nu) \end{aligned}$$

where n_1 and n_2 refer to the number of particles per unit volume in the lower and upper states, respectively. These relations were derived for discrete, bound states, of course. In the synchrotron case, we want to consider *continuum* states. But we can think of these states as being discrete if we break up the classical phase space into elements of volume h^3. Then, we sum over all states which absorb at frequency ν; namely, all states such that $E_1 = E_2 - h\nu$:

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_2} \left[n(E_2 - h\nu)\, B_{12} - n(E_2)\, B_{21}\right] \phi(\nu) \ .$$

Note that the power emitted by a single electron is

$$P(\nu, E) = 4\pi\, \frac{j_\nu}{n_2} = h\nu\, A_{21}\, \phi(\nu) = h\nu \left(\frac{2h\nu^3}{c^2}\right) B_{21}\, \phi(\nu)$$

where the Einstein relation $A_{21} = (2h\nu^3/c^2)B_{21}$ has been invoked. Also, recall that $B_{12}/B_{21} = g_2/g_1$. We now have

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \sum_{E_2} \left[\frac{g(E_2)}{g(E_2 - h\nu)}\, n(E_2 - h\nu) - n(E_2)\right] P(\nu, E_2) \ .$$

Next, we want to replace this sum over what are formally discrete states with an integral over *continuum* states. To do this, we identify $n(E)$ with the local density of states in phase space and $g(E)$ with the corresponding statistical weight. Thus,

$$\begin{aligned} n(E) &\Longrightarrow N(E)\, dE \ , \\ g(E) &\Longrightarrow E^2 \ \text{ or } \ p^2c^2 \end{aligned}$$

for relativistic, isotropic velocities. Then,

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \int_0^\infty E^2 \left[\frac{N(E - h\nu)}{(E - h\nu)^2} - \frac{N(E)}{E^2}\right] P(\nu, E)\, dE \ .$$

If we make the classical assumption that $h\nu \ll E$, then we can identify $-h\nu$ with ΔE and then the term in square brackets becomes $-h\nu\, \Delta(N/E^2)/\Delta E$:

$$\alpha_\nu = -\frac{c^2}{8\pi\nu^2} \int_0^\infty P(\nu, E)\, E^2\, \frac{d}{dE}\left[\frac{N(E)}{E^2}\right] dE \ .$$

At first sight, it looks as though this integral could be made *negative* by choosing $N(E)$ properly. If so, one would have a *synchrotron maser* wherein stimulated emission overpowers absorption.

However, the text provides a proof that this can never happen in a vacuum where the velocity of light is always c.

We now want to stick in a power-law distribution for $N(E)$ and the spectrum for a single electron $P(\nu,E)$. The text gives the expression for α_ν in the case where $P(\nu,E)$ is due to a uniform magnetic field. But this is not really consistent because some of our relations are valid only for isotropic velocity distributions or tangled magnetic fields. Ginzburg & Syrovatskii deal with the tangled field case. They find

$$\begin{aligned} j_\nu &= a(p)\left(\frac{e^3}{mc^2}\right)\left(\frac{3e}{4\pi m^3c^5}\right)^{(p-1)/2} K\,B^{(p+1)/2}\,\nu^{(1-p)/2}\,, \\ \alpha_\nu &= b(p)\left(\frac{e^3}{2\pi m}\right)\left(\frac{3e}{2\pi m^3c^5}\right)^{p/2} K\,B^{(p+2)/2}\,\nu^{-(p+4)/2}\,, \end{aligned}$$

where the coefficients $a(p)$ and $b(p)$ for three values of p are given in the following table:

p	$a(p)$	$b(p)$
1	0.283	0.960
2	0.103	0.700
3	0.0742	0.650

Actually, the frequency dependence for α_ν is the same for the uniform and tangled field cases. Only the coefficients are different. In fact, one can get the correct frequency dependence by assuming that the electrons radiate only at ν_m as we did earlier for j_ν. (This is a good student exercise!) Note how the absorption coefficient gets very small at high frequency but very large at low ν.

We can now easily predict what the observed flux spectrum will look like in the optically thick and thin limits. In the *optically thick* case, we know that

$$F_\nu \propto I_\nu \propto S_\nu = \frac{j_\nu}{\alpha_\nu} = s(p)\,B^{-1/2}\,\nu^{5/2}$$

where

$$s(p) \equiv \frac{a(p)}{b(p)}\left(\frac{2\pi}{c^2}\right)\left(\frac{1}{2}\right)^{(p-1)/2}\left(\frac{3e}{2\pi m^3c^5}\right)^{-1/2}\,.$$

Note that F_ν decreases at a given ν with increasing B! Why does this happen?

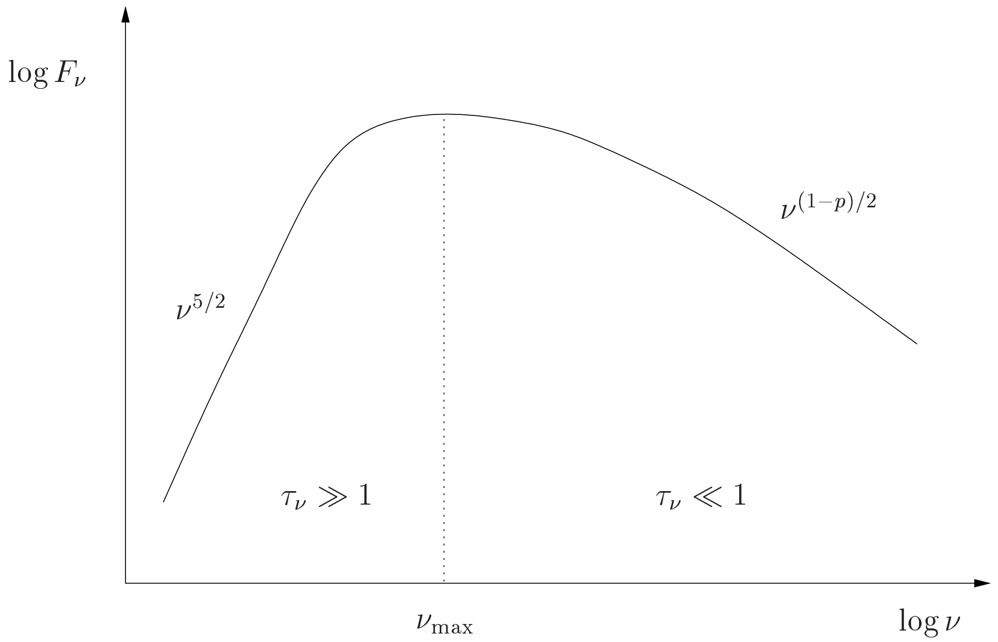
When the gas is *optically thin* to synchrotron radiation, on the other hand,

$$F_\nu \propto I_\nu \propto S_\nu\tau_\nu \propto j_\nu \propto B^{(p+1)/2}\,\nu^{(1-p)/2}\,.$$

The turn-over frequency $\nu_{\rm max}$ is normally too low to be seen in most astrophysical situations unless the density is very high or, equivalently, the brightness temperature is very high. For example, for the Crab nebula, $\nu_{\rm max} \sim 1\,$MHz which is well *below* the FM band! This is much different than the turn-over frequency for the thermal bremsstrahlung spectrum of H II regions. Basically, relativistic electrons don't readily absorb photons, especially in weak magnetic fields.

Nevertheless, in some extragalactic sources (such as quasars), K and B are large enough to produce a turnover at a sufficiently high frequency (1 GHz) to be observed.

Combining both limiting cases, we can see what the spectrum must look like:



Obviously, the slope of the optically thick part does not depend on p.

It is easy to make a crude estimate of the brightness temperature in the optically thick limit if we assume that the relativistic particles radiate as efficiently as those responsible for thermal emission. Remember that, in the case of thermal emission from compact HII regions,

$$kT_b \longrightarrow kT_e \;\sim\; \frac{1}{2}mv^2 \;\sim\; E_{\rm kin} \quad \text{when} \quad \tau_\nu \gg 1 \quad \text{and} \quad T_e \sim 10^4\,^\circ\text{K} \ .$$

If we apply the same reasoning to the synchrotron case, then we would have

$$kT_b \sim E_{\rm kin} = (\gamma - 1)mc^2 \qquad (\text{when } \tau_\nu \geq 1) \ .$$

Therefore,

$$T_b \sim \frac{\gamma mc^2}{k} = \frac{10^3 \times 0.51 \times 10^6 \text{ ev}}{8.6 \times 10^{-5} \text{ ev }^\circ\text{K}^{-1}} = 10^{13}\,^\circ\text{K} \qquad (\gamma = 10^3) \ .$$

Thus, in the optically thick limit, we expect to see brightness temperatures of order $10^{13}\,^\circ$K. But, for galactic sources, T_b is no higher that $10^{12}\,^\circ$K at radio wavelengths! Hence, the gas must be optically thin at these frequencies which it is because we normally observe at frequencies well above the peak.

Another reason that our estimate is higher than the observed limit is that relativistic particles actually radiate much less efficiently than thermal particles by as much as a factor 10. A better way to estimate the brightness temperature for a particular source for which you know the intensity (or flux) is to use the Rayleigh-Jeans law: $T_b \simeq (c^2/2k\nu^2)\, I_\nu$. For a strong source, this relation would typically give a result closer to $10^{12}\,^\circ$K.

Even in quasars and active galactic nuclei, T_b can't reach $10^{13}\,^\circ$K due to limitations imposed by Inverse Compton scattering (see later discussion in §5.3). Thus, even though some compact sources do show the turn-over, T_b is always less than $10^{12}\,^\circ$K.

4.6 CASE STUDY: Equipartition in Supernova Remnants

It is interesting to note that in at least some synchrotron sources (e.g., the galactic disk, supernova remnants), there appears to be an equipartition between the magnetic field and the relativistic particles. This indicates an intimate coupling between the two components and, indeed, the nature of the coupling is obvious. Large magnetic field energies lead to the acceleration of relativistic particles and, at the same time, large particle energies can generate magnetic fields. But neither can get too strong without damping the other.

It is easy to show that for a given synchrotron luminosity, a source has a *minimum* total energy near equipartition. In fact, if one assumes that the field in a particular source has the equipartition value for the observed luminosity, one can in some cases estimate what that value is along with the total energy of the system.

(a) Particle Energy:

$$\begin{aligned} W_e &= V \int_{E_1}^{E_2} E\, N(E)\, dE \;=\; V \int_{E_1}^{E_2} K\, E^{1-p}\, dE \\ &= \frac{KV}{(p-2)} \left(E_1^{2-p} - E_2^{2-p} \right) \qquad (p \neq 2)\,. \end{aligned}$$

Then,

$$W_p = W_{\text{particle}} = W_{\text{proton}} + W_e = (1+k)\, W_e$$

where k, the ratio of proton and electron energies, is left as a free parameter.

We can express KV in terms of the luminosity L as follows:

$$L = 4\pi V \int_{\nu_1}^{\nu_2} j_\nu\, d\nu = C_1\, K\, V\, B^{(p+1)/2} \left[{\nu_2}^{(3-p)/2} - {\nu_1}^{(3-p)/2} \right]$$

where ν_1 and ν_2 are critical frequencies corresponding to E_1 and E_2:

$$E_1 = C_2 \left(\frac{\nu_1}{B} \right)^{1/2} \quad \text{and} \quad E_2 = C_2 \left(\frac{\nu_2}{B} \right)^{1/2} \,.$$

The coefficients C_1 and C_2 are constants that follow from the equations on pages 64 and 70.

Now solve for KV above and then substitute into the expression for W_p. One finds

$$W_p = C_3\, B^{-3/2}$$

where

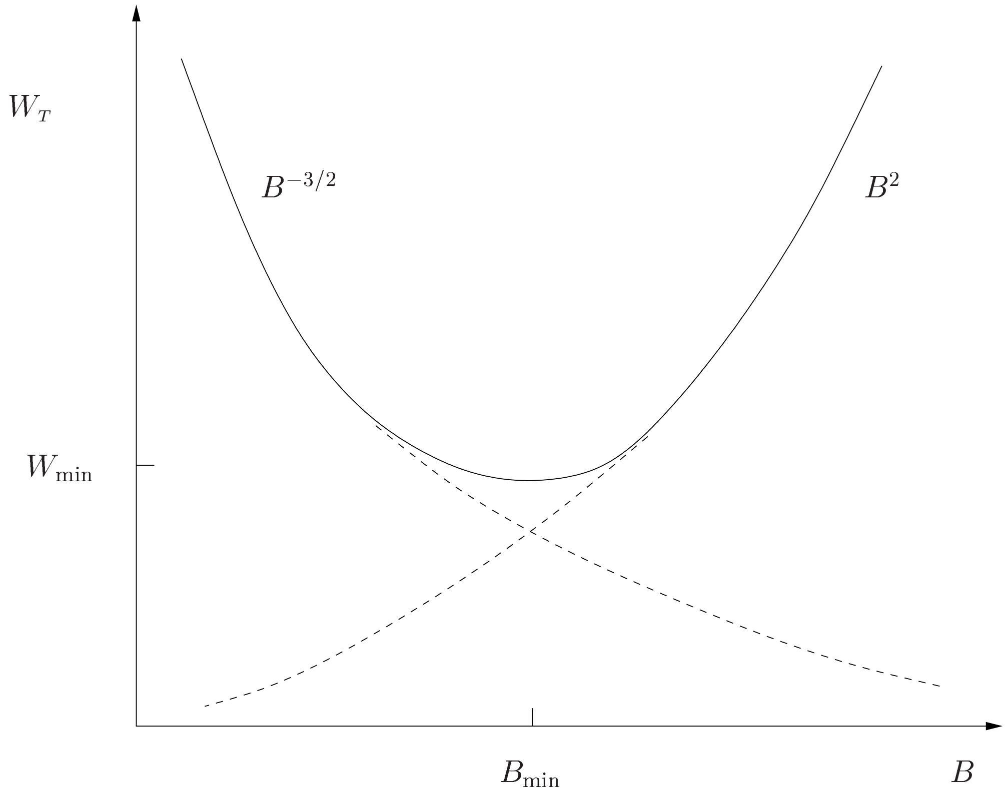
$$C_3 \equiv 5.75 \times 10^{11} (1+k)\, L \left(\frac{3-p}{2-p} \right) \left[\frac{\nu_2^{(2-p)/2} - \nu_1^{(2-p)/2}}{\nu_2^{(3-p)/2} - \nu_1^{(3-p)/2}} \right] \,.$$

(b) Magnetic Energy

$$W_{\text{M}} = \frac{B^2 V}{8\pi} = C_4\, B^2 \qquad \left(C_4 = \frac{1}{6} R^3 \right)\,.$$

Thus,

$$W_{\text{T}} = W_p + W_{\text{M}} = C_3\, B^{-3/2} + C_4\, B^2 \,.$$



We can easily find $B_{\min}$:

$$\left(\frac{dW_{\rm T}}{dB}\right)_{B_{\min}} = -\frac{3}{2}\,C_3\,B_{\min}^{-5/2} + 2C_4\,B_{\min} = 0\ .$$

Clearly,

$$\frac{3}{2}\,C_3\,B_{\min}^{-3/2} = 2C_4\,B_{\min}^2$$

or

$$\frac{3}{2}\,W_p = 2W_{\rm M} \qquad (B = B_{\min})\,.$$

Therefore, when $B = B_{\min}$ and the total energy is a minimum,

$$W_p = \frac{4}{3}\,W_{\rm M} \quad \text{(approximate equipartition)}\,.$$

We can also write

$$\begin{aligned} B_{\min} &= \left(\frac{3C_3}{4C_4}\right)^{2/7} \ \propto\ (1+k)^{2/7}\,R^{-6/7}\,L^{2/7}\ , \\ W_{\min} &= W_p + W_{\rm M} \ =\ \frac{7}{3}W_{\rm M} \ =\ \frac{7}{3}\left(\frac{B_{\min}^2}{8\pi}\right)V \ \propto\ (1+k)^{4/7}\,R^{9/7}\,L^{4/7}\ . \end{aligned}$$

Pacholczyk tabulates the missing coefficients here as functions of p, ν_1, and ν_2 (p233 of *Radio Astrophysics*). One generally assumes $\nu_1 = 10^7\,$Hz and $\nu_2 = 10^{11}$ or $10^{12}\,$Hz (i.e., E in the range 10^8 to $10^{11}\,$ev).

There are various forms of the equation for $B_{\min}$. One useful one is offered by G. Miley in *Ann. Rev.*, **18**, 165, 1980. It involves the surface brightness $(F_\nu/\theta_x\theta_y)$ as follows:

$$B_{\rm min} = 5.69 \times 10^{-5} \left[\frac{2(1+k)\, F_\nu\, \nu^{(p-1)/2}\, (1+z)^{(p+5)/2}\, (\nu_2^{(2-p)/2} - \nu_1^{(2-p)/2})}{\eta\,(2-p)\,\theta_x\theta_y\, S\, \sin^{3/2}\phi} \right]^{2/7} \text{gauss}$$

where

$$\begin{aligned}
\eta &\equiv \text{filling factor}\\
z &\equiv \text{redshift}\\
\theta_x,\, \theta_y &\equiv \text{major and minor axes of cross-section in arcsec}\\
S &\equiv \text{length of source along line of sight in kpc}\\
\phi &\equiv \text{angle of } \boldsymbol{B} \text{ with respect to line of sight}\\
F_\nu &\equiv \text{flux at frequency } \nu \text{ in GHz}\\
\nu_1,\, \nu_2 &\equiv \text{frequency limits in GHz}
\end{aligned}$$

APPLICATION TO THE CRAB NEBULA

Properties:

$$\begin{aligned}
\alpha &= (1-p)/2 = -0.3 \qquad (\text{for } p = 1.6 < p_{\rm CR})\\
F_\nu &= 1000\,\text{Jy at } \nu = 1\,\text{GHz}\\
\text{Jy} &= \text{Jansky} = 10^{-26}\,\text{W m}^{-2}\,\text{Hz}^{-1} = 10^{-23}\,\text{ergs s}^{-1}\,\text{cm}^{-2}\,\text{Hz}^{-1}\\
D &= 2\,\text{kpc}\\
\theta_x &= 2\,\text{arcmin} = 120'' = \text{b/D}\\
\theta_y &= 3\,\text{arcmin} = 180'' = \text{a/D}\\
S &= \frac{120''}{2\times 10^{5''}} \times 2\,\text{kpc} = 1.2\,\text{pc}
\end{aligned}$$

We assume here a prolate spheroid ($a > b = c$) for which $V = \frac{4}{3}\pi a b^2$. Note that the spectral index α for the Crab is atypically small; most supernova remnants have steeper slopes (-0.6 to -0.8) which may indicate that we are seeing the spectra *above* the break as one might expect for older remnants. In any case, the steeper slopes are consistent with the cosmic-ray energy distribution.

Energy:

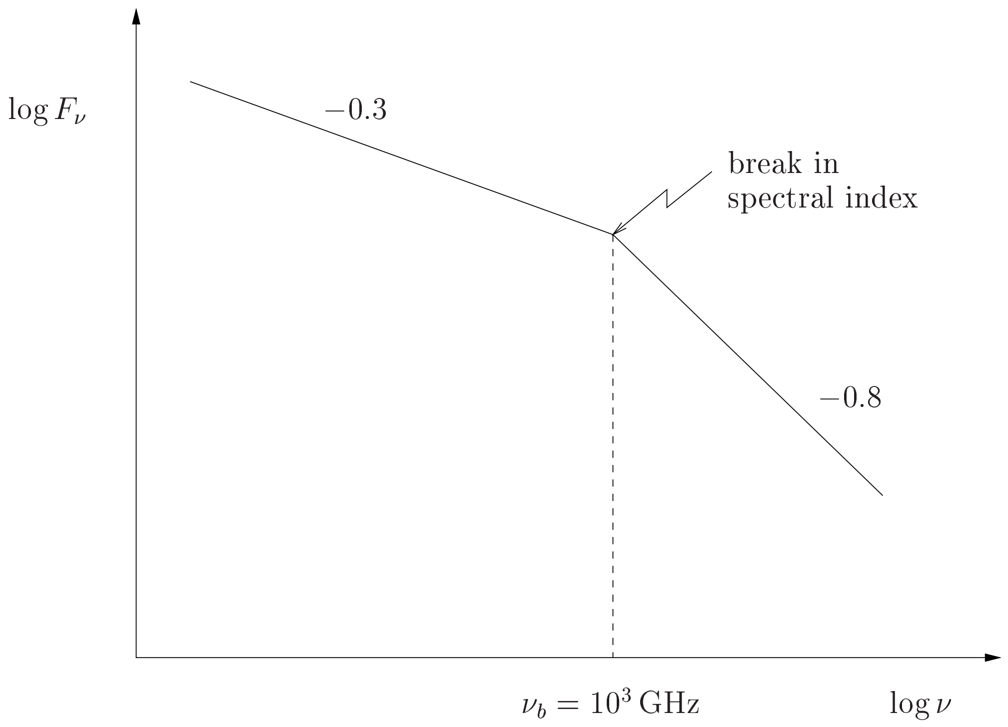
Use Miley's expression with $z = 0$, $k = 1$, and $\eta = 1$. Then the field at minimum energy is

$$B_{\rm min} = 5.69 \times 10^{-5} \left[\frac{4(1000)(1)^{0.3}}{(120)(180)(0.0012)} \left(\frac{100^{0.2} - 0.01^{0.2}}{0.4} \right) \right]^{2/7} = 3.86 \times 10^{-4}\ \text{gauss}\ ,$$

and the corresponding minimum energy is

$$W_{\rm min} = 0.0928\, B_{\rm min}^2\, V = 3.0 \times 10^{48}\,\text{ergs}\ .$$

Spectrum:



Note that there is no sharp cutoff here due to continuous injection of particles.

Can we interpret the break that we do see in terms of synchrotron losses? Suppose the ignition turned on at $t = 0$ (supernova!) and continued through to the current time (T), the age of the nebula:

(a) For energies for which $t_{1/2} \gg T$, the spectrum should look like

$$N(E) \simeq \phi(E)\,T$$

where $\phi(E)$ is the injection rate equal to $\phi_o E^{-p}$.

(b) On the other hand, at high E where $t_{1/2} \ll T$, then

$$N(E) \simeq \phi(E)\,t_{1/2}\ ;$$

i.e., equilibrium between injection and radiation losses.

But, $t_{1/2} \propto E^{-1}$ which means that $N(E) \propto E^{-p-1}$ at high E. Thus, the slope of the energy spectrum increases by unity and so the spectral index should steepen by 0.5 *as observed*! Therefore, we can interpret the break frequency ν_b to be the point where $t_{1/2} \simeq T$. That is,

$$T = \frac{5.1 \times 10^8}{B_\perp^2\,\gamma_b}$$

where γ_b is the value of γ at the break frequency. Setting $\nu_b = \nu_m = 1.26 \times 10^6\,B_\perp\,\gamma_b^2$, we obtain

$$T = \frac{5.72 \times 10^{11}}{B_\perp^{3/2}\nu_b^{1/2}} = \frac{5.72 \times 10^{11}}{(3.9 \times 10^{-4})^{3/2}(10^{12})^{1/2}} = 7.4 \times 10^{10}\,\mathrm{sec} \simeq 2500\,\mathrm{yr}\ .$$

This is twice the estimated age of the Crab Nebula; so our argument is at best only consistent. Remember that our analysis is not rigorous. For example, we have assumed throughout a one-to-one correspondence between frequency and energy; strictly speaking, such a direct relationship is only approximate. Also, this young nebula may not yet have reached equilibrium.

Note that the Crab pulsar injects about $10^{38}\,\mathrm{ergs\,s^{-1}}$. Hence, in 1000 yr, the total energy injected should be

$$10^{38} \times 3 \times 10^{10} = 3 \times 10^{48}\,\mathrm{erg}$$

which is just the equipartition energy that we estimated.

We could carry this application even further because we now have enough information to estimate K and, hence, the density of the energetic particles. This would permit an estimate of the optical depth τ_ν and, in particular, the frequency at which the spectrum turns over. Another interesting property that could be checked is the ratio of radiation and magnetic energies. That is, what is $R = W_{\mathrm{R}}/W_{\mathrm{M}}$ for the Crab nebula? It should be much less than unity (see §5.4). This is a good problem for a graduate student!